These notes are meant to supplement a presentation given on November 26th, 2021, for MAT477, based on Chapter 22 of the textbook *Fourier Analysis on Finite Groups and Applications* by Audrey Terras. Many ideas of this talk are inspired by and brilliantly explained to me by my friend Balazs Nemeth during a private lecture he gave in the summer of 2021.

1 Get Motivated

Given a graph, we can construct various different matrices and/or operators on it. These operators are worth studying as they often reflect and model real-life problems. Madhav constructed a few and demonstrated some usages of them a few weeks ago. Studying specifically the eigenvalues, or more generally the spectrum, of these operators will allow us to more concretely and easily manage these operators, which could sometimes be too complicated or difficult to work with.

But we wonder, just how powerful are these eigenvalues? Just how much information can the spectrum provide for us? One specific question we might ask is whether the spectrum completely determines the graph. To narrow down our investigation, we shall only consider this for the adjacency operator on finite graphs.

(Side note: in infinite dimensional situations, the spectrum of an operator A is defined as all λ such that $A - \lambda I$ does not have a bounded inverse. This may include values which are not eigenvalues, which are λ such that $A - \lambda I$ is not invertible.)

In this talk I first present the necessary graph theoretical definitions, then define and introduce representation theoretical tools such as characters and the fourier transform (yes, this talk is more like a talk on the application of representation theory on another field of math, haha. . .). Using these, I demonstrate a construction of isospectral but nonisomorphic graphs, and, finally, outline an example, the source of which I cite and hyperlink at the very end.

2 Groups and Graphs

Graph We denote a graph X as a pair of sets, $X = (V, E)$ where V is the set of vertices $v_1, ..., v_n$, and E is the set of edges, formally written as ordered (or unordered) pairs (v_i, v_j). Usually, if $(v_i, v_j) \in E$, we say they are connected by an edge, and it is denoted $v_i \sim v_j$.

Adjacency Operator Given a graph X on n vertices, its *adjacency matrix* is an $n \times n$ matrix $A_X \in \mathrm{M}_{n\times n}(\mathbb{C})$ whose ij-th entry is defined by

$$(A_X)_{ij} := \begin{cases} 1 \text{ if } v_i \sim v_j \\ 0 \text{ otherwise} \end{cases}$$

Note that with infinite graphs, the "matrix" is a little harder to think about. Thus, we can equivalently think about it as the *adjacency operator* which, given any function on the vertex set, $f : V \to \mathbb{C}$, we have

$$(A_X f)(v_i) := \sum_{v_i \sim v_j} f(v_j)$$

I.e. the adjacency operator operates on the space of functions on V. This operator definition will come in handy later.

Given that we know a good amount of facts about finite groups, especially with the help of representation theory, we want to reduce the graph theoretical problem to a group and representation theoretical problem by using groups to generate graphs. One way to do it is with Schreier graphs:

Schreier graph Given a subgroup Γ of a finite group G and a generating subset S of G which we call the edge set, the *Schreier graph* $X = X(G, \Gamma, S)$ has the cosets Γg, $g \in G$, as vertices. Two vertices Γg and Γgs are connected by a directed edge by any $s \in S$.

One last seemingly arbitrary definition for our setup:

Almost conjugate Two subgroups Γ_1, Γ_2 of G are *almost conjugate* if for every conjugacy class $\{g\} = \{xgx^{-1} \mid x \in G\}$, we have

$$|\{g\} \cap \Gamma_1| = |\{g\} \cap \Gamma_2|$$

Note that since the conjugacy classes of G partition G, almost conjugate subgroups have the same size, i.e. $|\Gamma_1| = |\Gamma_2|$.

Claim 2.1. *The Schreier graphs of two almost conjugate subgroups may not be isomorphic.*

I haven't been able to come up with a straightforward and convincing way to see this, but as you may verify with the example later, this is in fact true.

Now, the big important theorem that everything today banks on:

Theorem 2.2. *Suppose Γ_1, Γ_2 are almost conjugate subgroups of a finite group G. Then the Schreier graphs $X(G, \Gamma_i, S)$ are isopectral for any edge set S in G.*

3 Representation Theory

First, let V be an n-dimensional vector space, and recall the space of invertible "matrices":

$$\mathrm{GL}(V) = \{T : V \to V \,|\, T \text{ is linear and invertible}\}$$

Representation A (finite-dimensional) *representation* of the finite group G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. In plain words, feed any $g \in G$ into ρ, and $\rho(g)$ is an invertible matrix acting on the n-dimensional vector space V. Moreover, recall that a group homomorphism preserves the structure of the groups, which means we have, for any $g, h \in G$,

$$\rho(gh) = \rho(g)\rho(h)$$

Equivalent Representations Two representations ρ and π of a finite group G into $\mathrm{GL}(V)$ are said to be *equivalent* if there exists a matrix $T \in \mathrm{GL}(V)$ such that, for all $g \in G$, we have

$$T\rho(g)T^{-1} = \pi(g)$$

Character The *character* χ_ρ of a representation ρ of G is defined, for all $g \in G$, by

$$\chi_\rho(g) = \mathrm{Tr}\,\rho(g)$$

Properties of the character include:

1. If π, ρ are two representations, then $\chi_\pi = \chi_\rho$ if and only if π, ρ are equivalent representations.
2. The character χ_ρ is constant on each conjugacy class of G, i.e. for any fixed $g \in G$, we have $\chi_\rho(x^{-1}gx) = \chi_\rho(g)$ for any $x \in G$.

Fourier Transform Given any function $f : G \to \mathbb{C}$, define its *fourier transform* at ρ by

$$\hat{f}(\rho) = \sum_{x \in G} f(x)\rho(x^{-1})$$

Induced Representation Let G be a finite group and Γ a subgroup of G, and suppose $\pi : \Gamma \to \mathrm{GL}(W)$ is a representation of Γ. Then the *induced representation* from Γ up to G, denoted $\rho = \mathrm{Ind}_\Gamma^G \pi$ is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where

$$V = \{f : G \to W \,|\, f(\gamma g) = \pi(\gamma)f(g), \text{ for all } \gamma \in \Gamma,\ g \in G\}$$

On $f \in V$, the representation $\rho(g)$ is defined by

$$[\rho(g)f](x) = f(xg), \text{ for all } x, g \in G$$

Specifically, given any subgroup Γ of G, we can define $\rho = \mathrm{Ind}_\Gamma^G 1$ where 1 is the representation $1 : \Gamma \to \mathbb{C}$ that sends everything to 1. Following instructions from above, we end up with the space of functions

$$\begin{aligned} V &= \{f : G \to \mathbb{C} \,|\, f(\gamma g) = 1(\gamma) f(g), \text{ for all } \gamma \in \Gamma,\, g \in G\} \\ &= \{f : G \to \mathbb{C} \,|\, f(\gamma g) = f(g), \text{ for all } \gamma \in \Gamma,\, g \in G\} \\ &=: L^2(\Gamma \backslash G) \end{aligned}$$

Or simply known as the "Γ-invariant functions on G." Each function f in this space is constant on (right) cosets of Γ, and hence we can view them as functions on cosets of Γ in G.

Recall the definition of the adjacency operator from a page ago, which tells us how A operates on any function f on the set of vertices. Conveniently, since our Schreier graph has the cosets of Γ as its vertices, any $f \in L^2(\Gamma \backslash G)$ is actually just a function on the vertices of the Schreier graph generated by Γ. Thus. we find that

$$\begin{aligned} (Af)(x) &= \sum_{s \in S} f(xs) \\ &= \sum_{s \in S} [\rho(s) f](x) \end{aligned}$$

The first equality follows from the definition of adjacency operator summing over the vertices connected to x, which would be any xs with $s \in S$. The second equality follows from the definition of $\rho = \mathrm{Ind}_\Gamma^G 1$. Hence, we can write

$$A = \sum_{s \in S} \rho(s) \tag{A}$$

which is a neat way to write the adjacency operator/matrix in terms of representations induced by the specific subgroup.

4 Secret Sauce

Pre-trace Formula Given $\Gamma < G$ finite group, and induced representation $\rho = \mathrm{Ind}_\Gamma^G 1$, we have

$$\mathrm{Tr} \hat{f}(\rho) = \sum_{x \in \Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1} \gamma x)$$

To prove this, we need some more theorems and results about representation theory that I didn't state. Since I can't cover that in this talk, for now, just trust.

Choose $f = \delta_{\{g\}} : G \to \mathbb{C}$. Plug this into the pre-trace formula above to get

$$\mathrm{Tr} \hat{\delta}_{\{g\}}(\rho) = \sum_{x \in \Gamma \backslash G} \sum_{\gamma \in \Gamma} \delta_{\{g\}}(x^{-1} \gamma x)$$

Denote $\delta_{\{g\}}$ by simply δ. We compute each side of the formula.

Left-Hand Side:

$$\begin{aligned} \mathrm{Tr}(\hat{f}(\rho)) &= \mathrm{Tr}\Bigg(\sum_{x \in G} \underbrace{\delta(x)}_{\text{constant}} \underbrace{\rho(x^{-1})}_{\text{matrix}} \Bigg) \\ &= \sum_{x \in G} \delta(x) \mathrm{Tr}(\rho(x^{-1})) \\ &= \sum_{x \in G} \delta(x) \chi_\rho(x^{-1}) \\ &= |\{g\}| \chi_\rho(g^{-1}) \end{aligned} \tag{L}$$

Right-Hand Side:

$$\begin{aligned}
\sum_{x\in\Gamma\backslash G}\sum_{\gamma\in\Gamma}\delta(x^{-1}\gamma x) &= \sum_{x\in\Gamma\backslash G}\sum_{\gamma\in\Gamma}\delta(\gamma) \\
&= \sum_{x\in\Gamma\backslash G} |\{g\}\cap\Gamma| \\
&= \frac{|G|}{|\Gamma|}|\{g\}\cap\Gamma| \qquad \text{(R)}
\end{aligned}$$

Put it together: Recall that if $C_G(g)$ is the centralizer of g, then we have $|\{g\}| = |G|/|C_G(g)|$. Thus, since (L) = (R), we can write:

$$\frac{|G|}{|C_G(g)|}\chi_\rho(g^{-1}) = \frac{|G|}{|\Gamma|}|\{g\}\cap\Gamma|$$

Rearranging gives:

$$\chi_\rho(g^{-1}) = \frac{|C_G(g)||\{g\}\cap\Gamma|}{|\Gamma|}$$

Finally, everything comes together. Recall that almost conjugate subgroups Γ_1, Γ_2 satisfies $|\{g\}\cap\Gamma_1| = |\{g\}\cap\Gamma_2|$ and $|\Gamma_1| = |\Gamma_2|$. Thus, we have:

$$\begin{aligned}
\frac{|C_G(g)||\{g\}\cap\Gamma_1|}{|\Gamma_1|} &= \frac{|C_G(g)||\{g\}\cap\Gamma_2|}{|\Gamma_2|} \\
\chi_{\rho_1}(g^{-1}) &= \chi_{\rho_2}(g^{-1})
\end{aligned}$$

which shows that ρ_1, ρ_2 are equivalent, meaning there exists some $T \in \mathrm{GL}(V)$ such that $T^{-1}\rho_1(x)T = \rho_2(x)$, for all $x \in G$.

Also recall we can write the adjacency operator of a Schreier graph using induced representations as in (A), so

$$A_1 = \sum_{s\in S}\rho_1(s) \qquad\qquad A_2 = \sum_{s\in S}\rho_2(s)$$

Hence, we obtain:

$$\begin{aligned}
A_2 &= \sum_{s\in S}(T^{-1}\rho_1(s)T) \\
&= T^{-1}\sum_{s\in S}(\rho_1(s))T \\
&= T^{-1}A_1T
\end{aligned}$$

By linear algebra, we have that A_1, A_2 are similar matrices, and hence have the same spectrum. Yay!

5 A Non-Satisfying Example

Consider the following matrix group:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in (\mathbb{Z}/8\mathbb{Z})^\times,\, b \in \mathbb{Z}/8\mathbb{Z} \right\}$$

This group has order $|G| = 32$. Given any element of G, conjugating by another looks like

$$\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & (1-a)d + bc \\ 0 & 1 \end{pmatrix}$$

From this, observe that the $(1,1)$-th entry (top left) is constant, and that the parity of the $(1,2)$-th (top right) is also constant, as $a, c \in (\mathbb{Z}/8\mathbb{Z})^\times$ are always odd. Also, $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ is in the centre of G. From here, we can write down and conclude that there are 10 conjugacy classes in G, and their representatives are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 0 & 1 \end{pmatrix}.$$

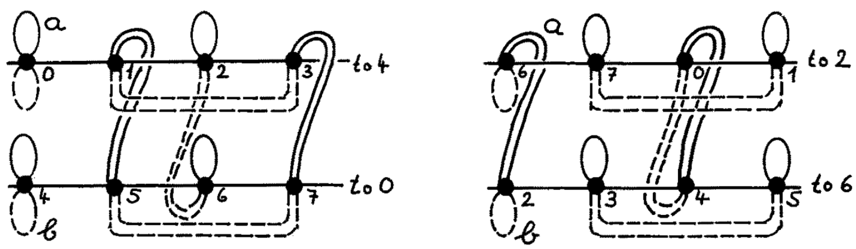
Then, we can choose our two almost conjugate subgroups to be:

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
$$\Gamma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Now a generating set we can choose is

$$S = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

Then, choosing coset representatives of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ gives the following Schreier graphs:



For more information, read Peter Buser's 1988 paper, Cayley graphs and planar isospectral domains. There are a few more interesting examples in there, and he more thoroughly goes through the process of building these isospectral nonisomorphic graphs.