These notes are meant to supplement a presentation given on October 1st, 2021, for MAT477, based on the paper, 'The Geometry of Musical Chords," by Dmitri Tymoczko. Below are some key definitions, extra explanation, and hopefully clarifying pictures.

1 Building Block Definitions

Chord A *chord* is a multiset of simultaneously occuring *notes*, which is a pitch or pitch class. We can order these notes by pitch, but the *quality* of the chord is more important than its order, so we simply consider an unordered multiset as opposed to something which is ordered.

Pitch We model pitches as real numbers. As a sound perceived by human ears, a *pitch* has a specific frequency in Hertz. Given the frequency f of a sound, its pitch in the *pitch space* (which we write as $\mathbb{R}$) can be modeled logarithmically by:

$$p(f) = 69 + 12\log_2(f/440)$$

This sends the 12 named notes to integers.

Pitch Class We define a *pitch class* by identifying the pitches in $\mathbb{R}$ via $p \sim p + 12$, so that the *pitch class space* is $\mathbb{R}/12\mathbb{Z}$.

Pitch Name: Letter to Integer Correspondence

C	0
$\mathrm{C}^\sharp = \mathrm{D}^\flat$	1
D	2
$\mathrm{D}^\sharp = \mathrm{E}^\flat$	3
E	4
F	5
$\mathrm{F}^\sharp = \mathrm{G}^\flat$	6
G	7
$\mathrm{G}^\sharp = \mathrm{A}^\flat$	8
A	9
$\mathrm{A}^\sharp = \mathrm{B}^\flat$	$10 =: t$
B	$11 =: e$

Distance, Pitch The *distance* between two pitches p, q is the absolute value of their difference, $|q - p|$.

Distance, Pitch Class The *distance* between two pitch classes a, b is the smallest nonnegative real number x such that, if p is a pitch belonging to a pitch class a, then either $p + x$ or $p - x$ belongs to pitch class b. It is denoted $||b - a||_{12\mathbb{Z}}$.

2 Chord-Pitch Class Space $\mathbb{R}^n/12\mathbb{Z}^n/\mathcal{S}_n$

Simplex A *k-simplex* is the set

$$\{a_0v_0 + ... + a_kv_k \mid \sum a_i = 1 \text{ and } a_i \geq 0\}$$

where $v_i - v_0 \in \mathbb{R}^k$ are linearly independent for $1 \leq i \leq k$.

We make the following changes of variables:

$$\begin{aligned} s_0 &= 0 \\ s_1 &= s_0 + a_0 \\ s_2 &= s_1 + a_1 \\ &\vdots \\ s_{k+1} &= s_k + a_k = \sum a_i = 1 \end{aligned}$$

which shows that the set $\{(s_1, s_2, ..., s_n \,|\, 0 \le s_1 \le ... \le s_k \le 1\}$ represents a k-simplex as well. The points such that $s_i = s_{i+1}$ are exactly the facet of the simplex.

Fundamental Domain of $12\mathbb{Z}^n/\mathcal{S}_n$ in $\mathbb{R}^n$ The fundamental domain, described by

$$\{(x_1, ..., x_n) \,|\, x_1 \le ... \le x_n \le x_1 + 12\} \text{ and } 0 \le \sum x_i \le 12$$

is an n-dimensional prism whose base is an $(n-1)$-dimensional simplex.

3 Comparing Voice Leadings

For motivation, we generally want independent and bijective voice leadings. A voice leading is *independent* if it is not a transposition, i.e. if it does not look like $(a_1, ..., a_n) \to (a_1 + b, ..., a_n + b)$ for some fixed value b. We will see the reasoning in a short moment. A voice leading being *bijective* means exactly what we expect it to mean: injectivity and surjectivity guarantees that each note in the first chord moves uniquely to a note in the second chord, and each note in the second chord has a fiber of one element from the first chord. When we use the word "voice-leading," we usually have a fixed number of voices.

For example, soprano, alto, tenor, bass, is a standard four-part set-up of a choral piece of music. In this case, we are determining the movements of four-note chords, who live in $\mathbb{T}^4/\mathcal{S}_4$. By requiring that all voice leadings are bijective, we guarantee that the music will be able to keep flowing, without parts disappearing or spontaneously appearing. What we want to do with each of these four parts is to minimize individual movement. It is not only difficult, but also unpleasant, to listen to someone sing drastically different notes consecutively. Imagine someone singing "somewhere" from "somewhere over the rainbow," and going back and forth singing only those two notes - it is simply not melodic and enjoyable. The most intricate and beautiful pieces of music will have sensible melodies in each part, while still maintaining a functional harmony on the large scale. Therefore, given a list of harmonies, we want to arrange the notes in each chord according to their parts while minimizing the distance between consecutive notes.

Voice Leading Given two multisets $\{x_1, x_2, ..., x_m\}$ and $\{y_1, y_2, ..., y_n\}$, a *voice leading* between them is a multiset of ordered pairs (x_i, y_j), such that every member of each multiset is in some pair. We denote a voice leading with the notation $(x_1, x_2, ..., x_n) \to (y_1, y_2, ..., y_n)$, noting that elements x_i and y_j are not in any fixed order and the subscripts are there mainly to keep track of the mapping.

Displacement Multiset Let $(p_1, p_2, ..., p_n) \to (q_1, q_2, ..., q_n)$ be a voice leading between multisets of pitches. The *displacement multiset* of the voice leading is the multiset of distances $\{|q_i - p_i|\}$. Similarly, if $(a_1, a_2, ..., a_n) \to (b_1, b_2, ..., b_n)$ is a voice leading between multisets of pitch classes, then its *displacement multiset* is $\{||b_i - a_i||_{12\mathbb{Z}}\}$.

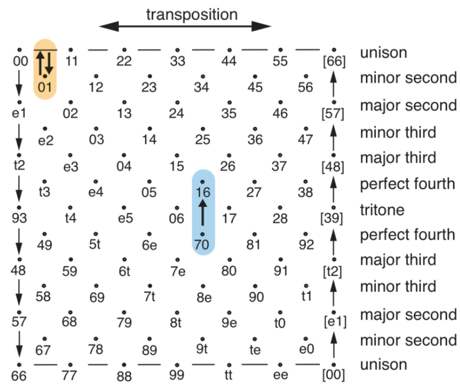
Distribution Constraint A relation "$\ge$" between multisets of nonnegative real numbers satisfies the *distribution constraint* if it satisfies:

$$\{x_1 + c, x_2, ..., x_n\} \ge \{x_1, x_2 + c, ..., x_n\} \ge \{x_1, x_2, ..., x_n\}, \text{ for } x_1 > x_2,\ c > 0$$

Claim 3.1. *If a method of comparing voice leadings obeys the distribution constraint, then there is a minimal bijective voice leading between any two chords that is crossing free.*

Claim 3.2. *Let A be a multiset $\{x_1, ..., x_n\}$, and "$\geq$" a total preorder satisfying the distribution constraint. Then For all x, the minimal bijective voice leading between A and $T_c(A)$, where $T_c(A) := \{x_1 + c, ..., x_n + c\}$, can be no smaller than the minimal bijective voice leading between E and $T_a(E)$ where E divides pitch-class space into n equal parts.*

To see a proof of these claims given by the original author, visit this link.



The orbifold $\mathbb{T}^2/\mathcal{S}_2$. C = 0, C♯ = 1, etc., with B♭ = t, and B = e. The left edge is identified with the right. The voice leadings (C, D♭)→(D♭, C) and (C, G)→(C♯, F♯) are shown; the first reflects off the singular boundary.

Figure 1: A useful diagram from Tymoczko's paper and his caption.

4 A Visualization of the Gluing in Two Dimensions

For a more concrete visualization, lets consider the 2-dimensional example given by Tymoczko. Figure 1 above shows the orbifold $\mathbb{T}^2/\mathcal{S}_2$. First, we have that $\mathbb{T}^2 = \mathbb{R}^2/12\mathbb{Z}^2$ can be represented by a square as shown below on the left hand side. By identifying the coordinates by the action of $\mathcal{S}_2$, we get the triangular fundamental domain on the right: Notice that our new equivalence relation does

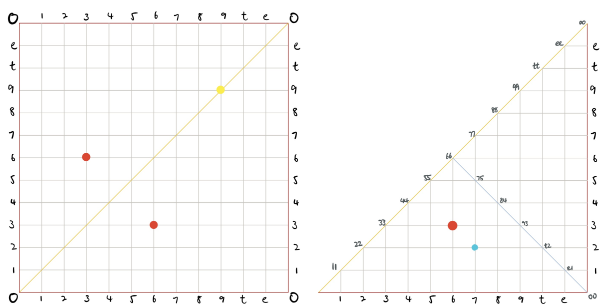


Figure 2: Gluing on the left, new quotient space on the right.

not glue points on the line $y = x$ to any other points, while points on either side of $y = x$ will be glued to the point which is its reflection from the line. Identifications, then, are made as in Figure 2: $(3, 6) \sim (6, 3)$, shown in red, while $(9, 9)$, shown in yellow, does not get glued anywhere else. Once again we can simplify this space by only considering the new quotient space, which is one of the two triangles of this red square, as shown in Figure 2. Noticing that the vertical red line on the right and the horizontal red line on the bottom are identical, we mentally "cut up" this triangle into two smaller isosceles triangles along the middle line segment which connects $\{6, 6\}$, denoted 66, to $\{0, 0\}$, denoted

00, through points 75, 84, 93, $t2$, and $e1$. Now we rotate the bottom triangle clockwise by 90 degrees, and “flip” it by 180 degrees. We can visualize this flip with the help of marking an extra point, 72, in blue, and tracking where it lands in the final alignment in Figure 3. Finally, we observe, on the left side of Figure 4, that the two opposite yellow edges, which go from 00 through 33 to 66 and from 66 through 99 to 00, are different. However, the two blue edges, which both go from 00 through 93 to 66, are identical. The tricky part, however, is that they are oriented with half-twist’s difference. This translates to the familiar square as shown on the right side of Figure 4.

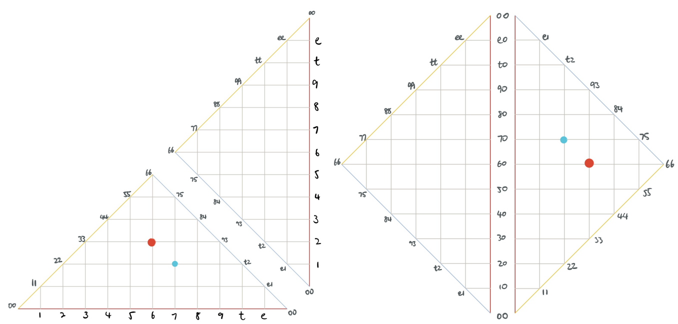


Figure 3: First treat it as two triangles, then align red edges.

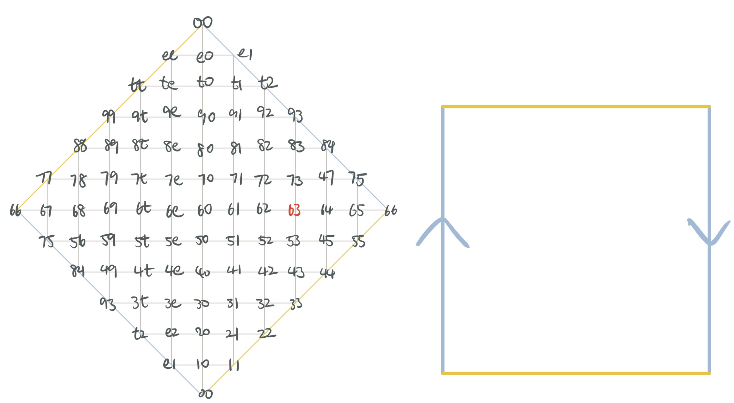


Figure 4: Each point labeled in the new square, $\mathbb{T}^2/\mathcal{S}_2$; identification of sides of new square.

References.

Tymoczko, Dmitri. "The Geometry of Musical Chords." *Science* 313: 72-74.