Introduction

Hello, C from the past. So you've decided to take MAT436: Introduction to Linear Operators. Good choice! Linear operators are super fun.

The way professor George Elliott runs this course, you'll have a lot of freedom to choose what you want to do with these 12 weeks. Here are some exercises that I did (or didn't do but wish I did), sourced from lectures, textbooks, and other explorations and curiosities. So if you are ever lost about what exercises to do, you can always look to this list for inspiration!! Of course, if you attend lectures, you should be getting plenty of exercises there too :)

The few texts that I consulted are listed at the end of this document in the references section, and the legend for reference: A for Axler, C for Conway, P for Pedersen (where "P E" means an exercise in Pedersen, and simply "P" means a result from the main chapter of the text).

1 Definitions to Know

It is extremely important to know and remember the definitions of these terms. Then, the more you work with them, the more you will gain intuition around what they *really* mean.

- Vector Space
- Normed Vector Space
- Banach Space
- Hilbert Space
- Dual Space
- Basis
- Adjoint Operator
- Algebra
- Operator Norm
- Bounded Operator
- Unitary Operator
- Normal Operator
- Projection Operator
- Positive Operator
- Finite-Rank Operator
- Compact Operator
- Index

2 Warm-Ups

Here is a list of exercises to get you started and familiar with the content of the course. You may want to do one or two of these each week before you dive into the deeper exercises, or do all of them at the beginning to solidify your understanding of foundational concepts.

Exercise 1. Make a new document or grab a fresh piece of paper. Look up definitions of each of the terms above and write down the definitions. If you run into more terms that you don't understand, look those up too and add them to the page. Continue adding to your masterlist of definitions as you go through these exercises.

Exercise 2. Show that multiplying matrices representing linear operators is equivalent to composing the linear operators.

Exercise 3. Show that the transpose of a linear operator on a vector space is an operator on the dual vector space.

Exercise 4. What are the linear operators in the 1-dimensional case?

Exercise 5. Show that a 1-dimensional vector space is isomorphic to the field itself.

Exercise 6. Consider the (ring) automorphisms of $\mathbb{C}$. Find one which takes i to $-i$.

Exercise 7. Consider the three equivalent definitions of a basis:

1. Minimal generating set.
2. Maximal linearly independent set.
3. Every vector in the space is uniquely a linear combination of elements in this set.

Why are they equivalent?

You may want to think about the equivalence and interdependencies of different linear algebra theorems. Try to prove the equivalence between the *well-definedness of number of elements in a basis* and the rank-nullity theorem. Which assumptions are you making about basis and "dimension"?

Exercise 8. Review the open mapping theorem and the closed graph theorem. Prove them?

Exercise 9. Recall the *winding number* of a closed curve from algebraic topology. Show that it is invariant under continuous deformation.

Exercise 10. Given 2 closed curves $S^1 \to \mathbb{C} \setminus \{0\}$, come up with ways to "merge" them together into a single curve. How is the winding number affected in each case?

Eventually, you might want to study Toeplitz operators, which is a starting point for understanding how the index of an operator is related to the index of a closed curve in a plane.

A note on notation/convention: if I don't clarify, assume V is a vector space, X is a normed or Banach space, and H is a Hilbert space.

3 Normed Spaces

Before we introduce further structure, it is helpful to get used to working with normed vector spaces, which, for me, bridged analysis and algebra. Pedersen's chapter 2.1 is a great resource to get familiar with definitions and basic results.

Exercise 11. If a vector space V is finitely generated (a linear span of a finite set), then a subspace $S \subset V$ is also finitely generated. You may want to consider the following:

(a) If V is finitely generated and $D \subset V$ is a subspace, then the quotient V/D is also finitely generated.

(b) Any subspace S of a vector space V is a quotient space V/D for some subspace $D \subset V$.

Exercise 12 (P 2.1.9)**.** Prove that every finite-dimensional subspace Y of a normed space X is a Banach space and hence closed in X; moreover, show that if $\dim(Y) = n$, then every linear isomorphism of $\mathbb{F}^n$ onto Y is a homeomorphism.

Exercise 13 (P E2.1.2)**.** Given an operator $T : X \to Y$ where Y is finitely generated, show that T is continuous iff $\ker T$ is closed in X.

Exercise 14 (P E2.1.3)**.** Show that the closed unit ball in a normed space X is compact iff X is finite-dimensional.

Exercise 15 (P E2.1.4)**.** Let Y and Z be closed subspaces of a Banach space X. Show that $Y + Z$ is closed if Z is finite-dimensional. (*Hint*: Consider the quotient map $Q : X \to X/Y$ and apply P 2.1.9 to $Q(Z)$, noting that $Z + Y = Q^{-1}(Q(Z))$.

Exercise 16 (P E2.1.5)**.** Show that ℓ^∞ is a Banach space and is not separable.

4 Dual Space, Inner Product

We now get familiar with the dual space as well as an inner product on a vector space. It is with the dual space that we can begin thinking of an adjoint operator. In the Hilbert Space case, rather than operating on the dual spaces, adjoint operators are operators on the same space, which leads to the interesting self-adjoint operators.

Exercise 17. Prove that an inner product induces a norm, and hence an inner product space is a normed space. This means that the definition of a Hilbert space as a *complete* inner product space makes sense.

Exercise 18. Suppose V is finite-dimensional. How does an inner product induce an isomorphism between V and its dual space V^*? What about the case when V is infinite-dimensional?

Exercise 19. Similarly to above, if you have an isomorphism $\Phi : V \to V^*$, can you recover an inner product on V?

Exercise 20. Suppose V is again finite-dimensional, and T a bounded linear operator on V with corresponding matrix A with respect to some basis $(e_1, ..., e_n)$. Consider the conjugate transpose matrix $\overline{A^T}$. Show that this is an operator on V^* and describe it. Check (A 3.99) for hint.

Exercise 21 (P 2.3.10)**.** Let $T : X \to Y$ be a bounded operators on normed spaces X, Y, show that the adjoint operator T^* is in $B(Y^*, X^*)$, i.e. $T^* : Y^* \to X^*$ is bounded, and that $||T^*|| = ||T||$.

Exercise 22. Explain why on a Hilbert space, we can think of the adjoint operator as an operator on the original space, instead of on the dual space.

Exercise 23 (P E2.3.4)**.** Show that a normed space X for which X^* is separable is itself separable.

Exercise 24 (P E2.3.5)**.** Show that a Banach space X is reflexive iff X^* is reflexive.

Exercise 25. On closedness, and orthogonal complements of Hilbert spaces:

(a) Give an example of a non-closed subspace.

(b) Compute the orthogonal complement of the space you gave above.

(c) Give an example of a non-closed subspace with $\{0\}$ as its orthogonal complement.

Now that we have an inner product and hence a sence of orthogonality, another toy we have is the orthogonal complement.

Exercise 26. For a Hilbert space H, take any $K \subset H$. Show that the orthogonal complement $K^\perp$ is a closed linear subspace, no matter what K is!

Exercise 27. Prove that if a subspace S of a normed space H is closed, then $(S^\perp)^\perp = S$.

Exercise 28. With the same S as above, is it true that $(S^*)^* = S$? What about the converse?

Exercise 29 (P E3.1.1)**.** Given n vectors $x_1, x_2, ..., x_n$ in the Hilbert space H, consider the $n \times n$ matrix $A = (a_{ij})$ given by $a_{ij} = (x_i|x_j)$. Show that the vectors are linearly independent iff $\det(A) \neq 0$. (Recall that $(\cdot|\cdot)$ is the inner product on H.)

Exercise 30 (P E3.2.23)**.** Define S in $B(\ell^2)$ by

$$(Sx)_1 = 0, \qquad (Sx)_{n+1} = x_n, \qquad x = (x_n) \in \ell^2$$

This is called the "unilateral shift" operator.

(a) Find S^*.

(b) Show that S has no eigenvalues, but that every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue for S^* with multiplicity 1.

(c) Show that none of the eigenvectors for S^* are orthogonal to each other.

Exercise 31 (P E3.1.5)**.** Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ and consider the Hilbert space $L^2(\mathbb{T})$ with respect to the Lebesgue measure on the circle. Show that the functions $e_n(z) = \frac{1}{\sqrt{2\pi}} z^n$, where $n \in \mathbb{Z}$, form an orthonormal basis for $L^2(\mathbb{T})$.

The last exercise here is particularly important, as it opens the gate to Toeplitz operators. We can define *the Hardy space* H^2 as the closed subspace of $L^2(\mathbb{T})$ spanned by the vectors e_n, for $n \geq 0$, as outlined in the exercise above.

Exercise 32 (P E3.3.14)**.** Let P be the projection from $L^2(\mathbb{T})$ onto H^2. We define the *Toeplitz* operator $T_f \in B(H^2)$, for each $f \in L^\infty(\mathbb{T})$, by

$$T_f(x) = P(fx), \, x \in H^2$$

Show that the map $f \mapsto T_f$ is linear and norm-decreasing ($L^\infty(\mathbb{T}) \to B(H^2)$, and that $T_f^* = T_{\overline{f}}$.

5 Self-Adjoint Niceness, Spectral Theorem

The bounded operators on a Hilbert space is a main object of interest. The various special types of bounded operators are very helpful to understand and work with. To begin, we familiarize ourselves with unitary, positive, and partial isometric operators by investigating the Polar Decomposition. Then, we have a few exercises on self-adjoint operators which eventually leads to the Spectral Theorem.

Exercise 33. *Polar Decomposition.*

1. Review the proof of (or prove) the finite-dimensional Polar Decomposition (A 7.45).
2. (P 3.2.11) Let T be a positive operator in $B(H)$. Prove that "the positive square root" $T^{1/2}$ which satisfied $(T^{1/2})^2 = T$ is well-defined and is a positive operator.
3. Review the proof of (or prove) the infinite-dimensional Polar Decomposition (P 3.2.17).
4. Prove that if T is invertible, the partial isometry in its Polar Decomposition is unitary.

Exercise 34. What are the self-adjoint operators in the 1-dimensional case?

Exercise 35. Come up with a few examples of self-adjoint operators on the common Hilbert spaces. Comment on their eigenvalues: do they exist? What are they?

Exercise 36. Suppose an operator T is invariant on some subspace $S \subset V$. Show:

1. T is not necessarily invariant on the complement of S.
2. But! If T is self-adjoint, then it *is* invariant on the complement.

Exercise 37. Given any bounded operator $T \in B(H)$, we can write $T = A + iB$ where A, B are self-adjoint.

Exercise 38. Remind yourself of the statement of the Spectral Theorem in finite-dimensional spaces. Guess what the Spectral Theorem is in infinite dimensions!

Exercise 39. *Fun with ellipsoids.* Given a finite-dimensional Hilbert Space, prove that ellipsoids, with respect to some orthonormal basis, are invariant under transformation by any invertible linear operator. Is this claim equivalent to the Spectral Theorem? Justify.

Exercise 40. As a warm-up to the "Polynomial Theorems", prove the following results:

(a) If $||x|| = 1$, then $-1 \leq (Ax|x) \leq 1$ iff $||A|| \leq 1$.

(b) If $-\varepsilon \leq p([-||A||, ||A||]) \leq \varepsilon$, then $-\varepsilon \leq p(A) \leq \varepsilon$ (i.e. $-\varepsilon \leq (p(A)x|x) \leq \varepsilon$).

(c) If $-\varepsilon \leq p([-||A||, ||A||]) \leq \varepsilon$, then $||p(A)|| \leq 4\varepsilon$.

(d) In fact, this stronger result holds: $||p(A)|| \leq \varepsilon$.

Exercise 41. *The Polynomial Theorem.* Suppose $A = A^* \in B(H)$ where H is a Hilbert space, and let $p(X) \in \mathbb{R}[X]$ be a polynomial. Then,

$$||p(A)|| \leq \sup_t |p(t)|$$

where t is a real number such that $-||A|| \leq t \leq ||A||$.

(a) Prove the *finite-dimensional case*, i.e. assume H is a finite-dimensional Hilbert space and prove the statement above.

(b) Read Israel Halperin's article in the American Mathematical Monthly, Apr 1964, and study his very cool and down-to-earth proof of this result: Suppose that A is a bounded self-adjoint operator on a Hilbert space H, and suppose:

i) $a, b \in \mathbb{R}$ and $a||x||^2 \leq (Ax|x) \leq a||x||^2$ for all x

ii) $p(X) \in \mathbb{R}[X]$ and $p(t) \geq 0$ for $a \leq b$.

Then, $p(A) \geq 0$.

(c) Prove the *infinite-dimensional* case, i.e. assume the Hilbert space H is infinite-dimensional. (*Hint:* Look at finite-dimensional invariant subspaces. Think sequences of projections in the strong operator topology.)

Exercise 42. What is a "multiplication operator"? Show that a multiplication operator is self-adjoint.

Exercise 43. *Definitely* not a trick question (/s): What does the Spectral Theorem say about multiplication operators?

Exercise 44. Look up the Spectral Theorem in infinite-dimensional Hilbert spaces; there are many versions and/or statements. Compare and contrast.

6 Compact Operators, Index

Exercise 45. Prove that T is compact iff $T = \lim_{n\to\infty} T_n$ where each T_n is a finite-rank operator.

Exercise 46. Prove that T is a finite-rank operator iff T^* is a finite-rank operator. Deduce that T is compact iff T^* is.

Exercise 47. Compare the Spectral Theorem for compact operators with the finite-dimensional one. Assuming the finite-dimensional Spectral Theorem, can you prove the infinite-dimensional compact version?

Exercise 48. There are two definitions of "cokernel" of an operators T on a Hilbert space H. As a vector space, the cokernel of an operator is

$$\operatorname{coker}(T) := H/\operatorname{Im}(T)$$

As a Hilbert space, only if $\operatorname{Im}T$ is closed, we can also define

$$\operatorname{coker}(T) := (\operatorname{Im}(T))^{\perp}$$

Show that they are equivalent on a Hilbert space.

Exercise 49. Given a Hilbert space H and a bounded operator $T : H \to H$, prove:

$$(\operatorname{Im}(T))^{\perp} = \ker(T^*)$$

Exercise 50. Show:

(a) The index of a direct sum is the sum of the indices.

(b) The index of a composition is the sum of the indices.

Exercise 51. More on index:

(a) Find a few examples of operators which don't have index. Can you "categorize" them into different kinds?

(b) Find operators with index 0 who don't live in a finitely generated space.

(c) Suppose $T \cong T|_{V_1} \oplus T|_{V_2}$ where V_1 is finitely generated and $T|_{V_2}$ is invertible in V_2. Show that the index of T is 0.

(d) Is every T with index 0 a direct sum like above?

Exercise 52. Given an operator T which has index, does its adjoint T^* have index? If so, what is it?

7 Banach Algebra, Atkinson's Theorem

Exercise 53 (C 2.1)**.** Show: if A is a Banach algebra with identity, and $x \in A$ is such that $||x-1|| < 1$, then x is invertible.

Exercise 54 (C 2.2)**.** Show: if A is a Banach algebra with identity, then the sets of left invertible elements, right invertible elements, and invertible elements are each an open set in A. Moreover, the map $a \mapsto a^{-1}$ is continuous.

Exercise 55 (C 2.6)**.** Show: if A is a Banach algebra and M is a proper closed ideal of A, then A/M is also a Banach algebra. If A has identity, then so does A/M.

Exercise 56. In what language was the original paper by Atkinson on "Atkinson's Theorem" written?

Let $B(H)$ denote the bounded operators on H, and $K(H)$ the compact and $F(H)$ the Fredholm.

Exercise 57. Show the following:

(a) $B(H)/K(H)$ is a Banach Algebra. (This is the Calkin Algebra.)

(b) the subset of $F(H)$ of all operators with index 0 is connected.

(c) the subset of $F(H)$ of all operators with index $n \in \mathbb{Z}$ is connected. Hence, taking index is a continuous map.

Exercise 58. Given a topological group G and a subgroup H, show that H is open iff G/H is discrete.

Exercise 59. Show that the invertible elements in $B(H)/K(H)$ is a topological group, and that the connected component to the identity element is an open subgroup.

Exercise 60. What are the conditions on a topological space which guarantees that you can only have countably many disjoint open subsets? For example, can you find uncountably many connected components in $B(H)$?

Exercise 61. Exercises (P E3.3.15-19), a series on Toeplitz operators, index, and winding number. Good luck!

Exercise 62. Exercises (P E4.1.17-19), more exercises involing the winding number.

Exercise 63. Consider the connected component of the identity element of $B(H)/K(H)$ as considered four exercises ago. Show that ("invertibles"/"connected to 1") $\cong \mathbb{Z}$. What does Atkinson's Theorem say about this?

References

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- Conway, John. (1990). *A First Course in Functional Analysis.* Springer.
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- Pedersen, Gert K. (1988). *Analysis Now.* Springer-Verlag.