Discovery of Incommensurable Numbers:

The Irrationality in Believing That This Discovery Was a Crisis to the Ancient Greeks

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The Ancient Greeks, specifically the Pythagoreans, have been known to be the first discoverers of irrational numbers. It is generally believed that this discovery caused a “crisis” in ancient Greek civilization, as they had previously come to know the world and its naturally occurring numbers to all be whole and proportional. While it is true that the Greeks had relied heavily on ratios between whole numbers, the discovery that not all numbers could be represented as ratios did not, in fact, cause a large historic disturbance that affected their mathematical development negatively, but only pushed their intellectual curiosity further. The discovery of irrational numbers initially caused a small disturbance, as mathematicians' social status, intellect, and knowledge base were all simultaneous challenged. However, because of its long lasting effects and inspiration, which we will explore in this essay, of the geometrization of algebra, restructuring of proportional theory, and advancements in number theory such as proving the fact that the real numbers are not discrete, this discovery is not “apocalyptic” after all and had overall positive effects on the development of mathematics historically and globally.

The earliest Greek Mathematics began with heavy Babylonian influences. The Ancient Babylonion Eastern civilizations had massive interest in arithmetic and algebra. For example, the famous artifact Plimpton 322 is a tablet dated back to the Babylonians, and shows that they had a very good understanding of Pythagorean triples (in modern understanding, numbers x,y,z such that $x^2 + y^2 = z^2$). While it definitely was not called “Pythagorean triples” for the Babylonians, as Pythagoras had not yet been born at this point, this certainly points to their interest in numerical properties, and demonstrates their ability to perform difficult arithmetic operations to obtain these large numbers. This interest in numbers had led the Ancient Greeks to start out this way as well, investigating numbers, the properties of different numbers, and the relationships between numbers.

As time went on, however, the ancient Greeks started to shift away from purely investigating properties of numbers and magnitudes, and started to emphasize rigor and reason. This could partially be attributed to the fact that they were a slave society, and social status was an important label for all. In Aristotle's (~350BC) influential work *Politics*, he states that a slave "is by nature […] capable of belonging to another (and that is why he does so belong), and […] participates in reasons so far as to apprehend it but not to possess it" (I.II.13). A slave's position is justified, along with other things, by the claim that the ability to reason implies intelligence, and since slaves "by nature" cannot reason, they are not intelligent enough, and would even benefit from being enslaved and ruled over by someone who can indeed use logic and reason, and who is hence capable of leadership. This emphasis on logic and reasons intensified both in society and in academia for mathematicians, and it soon became a necessity to logically support and/or prove every discovery, every theorem, and every construction. As members of society who were the most capable of logic and reason, being a mathematician in this era of Ancient Greece was a highly respected job. Thales of Miletus, in fact, was known to be the first to provide rigorous proofs for geometric theorems. He was also known to have travelled to Egypt to study, and hence brought back their geometric knowledge to Ancient Greece to study. By the time Pythagoras came around in 6th century BCE, there was already a large collection of mathematical proofs and constructions for the ancient greek mathematicians to work with.

The Pythagoreans especially loved whole numbers. The motto of the Pythagorean school was "all is number," which carries their belief that whole numbers are the basis and foundation of all numbers, and everything in the world could be considered in terms of wholes and their ratios. Many, if not most, of the proofs that came into existence up until this point used theory of proportion, which was in fact heavily based on the concept of commensurability. A commensurable number is one that can be presented as a ratio of two whole numbers - and the Greeks, especially the Pythagoreans, lived by the belief that every magnitude is commensurable. An incommensurable number, on the other hand, is one that cannot be represented as a ratio of two whole numbers, no matter how large or small the two numbers are. The discovery of incommensurable numbers, therefore, was a highly controversial and shocking event. It was so significant that the discoverer and arguably the first proof-provider of the existence of incommensurable numbers, Hippasus, was allegedly punished and drowned at sea for such a controversial discovery. One can only imagine the amount of stress, embarrassment, and despair mathematicians had felt when a large amount of previously valid proofs based on proportion theories suddenly became invalid. What did this say about their abilities to reason? Were they also incompetent to reason, as they believed for slaves? This also meant that the motto of the Pythagorean school did not hold anymore. We knew, then, that all is not number (i.e. wholes and their ratios), and that some quantities or magnitudes simply are just not able to be represented by whole number ratios and simple arithmetic. This shocking discovery led to many things; one can almost claim that it redirected the course that mathematics was taking. As number theory and proportions theory were largely based on commensurability, any proofs based on a theorem whose proof relied on commensurability also immediately became invalid, and therefore required new proofs. Two of the most significant changes worth noting, according to historian

Wilbur Knorr, were the geometrization of algebra, and the restructuring of proportion theory. Neither of these changes were surprising events.

After the existence of irrational numbers became well-known, the Ancient Greek society saw a heavy shift from arithmetic algebra to geometric algebra. Many algebraic properties, prior to the discovery, had already been represented in geometrical scenarios and using geometrical properties of lines, planes, shapes, and proportions. For example, the well-known algebraic property, $(a+b)^2 = a^2 + 2ab + b^2$, was proven using a large square that contains two smaller squares and 2 rectangles. But the existence of irrational numbers pushed these geometric proofs to the mainstream, simply because irrational numbers were extremely difficult, if not impossible, for the ancient Greeks to represent at this point, while a magnitude with an irrational value is easier to represent, understand, and work with. With their ancient number system which did not include an infinite decimal representation, writing out an irrational number such as $\sqrt{2}$ was not only impossible to represent precisely, but also hard to understand (Anglin & Lambek, 1995). Geometrically, on the other hand, the square root of 2 is exactly the hypotenuse of a right triangle with the other two sides having a magnitude of 1. This definition and representation of this number is accurate and precise, and provides lots of space for mathematicians to work with, as triangles and squares were already very familiar objects at this point.

The switch over to geometry from arithmetic, then, is justified. One cannot conclude that it was an escape into familiar territory out of fear of the incommensurables without considering the circumstances of the availability of mathematical tools to the Ancient Greeks. Additionally, to their credit, despite the shift into rigorous geometry, the ancient Greeks were still able to make good enough approximations of irrational numbers to properly engineer objects in real life

(Knorr, 1975, p. 308). For example, as explained by Anglin and Lambek in their 1995 book *The Heritage of Thales*, the value of the square root of 2 is approximated by the ancient greeks using the equation that we would modernly represent as $x^2 - 2y^2 = \pm 1$, where the precision of the approximation depends on the values we use for the variables.

With geometrical tools, many mathematical facts and theorems could still be proved without conflicting with incommensurability. This was made possible by Eudoxus, who first formally made the distinction between magnitudes and numbers. To Eudoxus, “numbers” referred to what we know today as whole numbers and their ratios, while “magnitudes” meant some entity that could be continuous instead of discrete (Kline, 1990, p.48). When numbers are used, actual numerical representations were required; when magnitudes of certain geometric shapes were involved, there is no requirement for a precise numerical representation of it, as the Greek had no ability to do so in the first place. In fact, geometrical representations of planar objects with specific magnitudes made possible that theorems were stated and proved with validity without replying on commensurability and without contradicting incommensurability. The geometry of triangle congruence, for example, is not based on previously known proportion theory, and therefore does not rely on the commensurability of the magnitudes of proportion (Knorr, p. 308). Geometry, in this sense, allowed the Ancient Greeks to represent irrationals in almost an abstract way, and to manipulate irrationals freely to obtain results without specific numerical values attached. Out of this also stemmed Eudoxus’ infamous Method of Exhaustion, which is one of the earliest methods that resembled calculus. Through Eudoxus’ genius work of the Method of Exhaustion and renewed theories of proportion that took into consideration incommensurable magnitudes, new sets of proofs were provided for the previously proven but then invalidated proofs of theorems and identities.

Yet another advancement in the discovery of incommensurability is its contribution to proving that all is not discrete but continuous. It was generally believed, especially by the Pythagoreans, that all quantities are discrete; that is, made of whole numbers, or ratios of whole numbers, no matter how small, but always able to be split up into portions. It had already been argued that not all number must be able to be divided into ratios of very large numbers for small precision, but disproving a proposition or claim does not prove the counterargument. In fact, while Zeno of Elea successfully came up with counterexamples and contradictive paradoxes as proofs for the common beliefs being wrong, he was unable to rigorously prove that not all measurements are discrete until the proof of the existence of irrational numbers. This undoubtedly positively proved his speculations correct, that one cannot divide every number into compositions of smaller fractions of whole numbers.

While the social stakes of their proofs being invalidated were high, the Greeks, contrary to popular belief, neither considered the discovery of incommensurables a “crisis” that needed resolution, nor completely gave up; rather, they soon accepted the new discovery, kept making progress on studying incommensurability, positively strived for new valid proofs for old theorems, and continued to make advances in different fields of mathematics despite this slight setback. In fact, in a sense, this was not a setback at all. The discovery of incommensurable numbers only expanded the knowledge of the ancient greek mathematicians, and allowed them a better understanding of the properties of numbers, and created better, more rigorous, and more valid proofs. One could argue that this discovery was inevitable, as mathematicians would naturally encounter numbers such as $\sqrt{2}$ and start to explore its properties and realize that it was not divisible into any nice whole numbers. Other areas of the world, such as India, had also discovered irrational numbers independently from the Greeks, further backing the argument that eventually, a civilization that is in the frontier of mathematics, would have discovered the existence of irrational numbers. We would not have geometry, algebra, number theory, and the distinction of discrete math, as opposed to the continuous, had the Greeks not discovered the incommensurables. In this sense, the "crisis" with which we often describe the discovery of irrationals is actually more accurately described as a historically significant revelation worth celebrating.

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