Contents

0 A Few Words (You Don't Have To Read These)

Hi, welcome to my crappy notes.

It is July 29th when I started this document, and we're already way past the flow stuff. But, the current problem set talks about Frobenius Theorem, so I better get this organized. Anything before this though... Maybe I'll never get a chance to type those out. :pensive:

The point of this document, originally, was to throw all the definitions and important results in one place without too much detail so I can easily access them - it's not easy to be a math student with a memory as shitty as mine. It seems, however, that I began making a lot of side comments for myself to explain the concepts, which made it not as skeletal as it was supposed to be. But the organizing and writing of these notes greatly helped me make sense of the math, so I am at peace with my infidelity to my original intentions.

In the formatting of this document, I stayed away from using the "Definition" environment (simply because there are way too many definitions), but rather just put the name of the object which is to be defined (or the property which is to be explained) in the title of the paragraph, and hence it is in boldface. This conveniently makes it pop more, so I can find it easier when frantically searching through my notes for a definition. Other environments such as theorem or proposition are all kept as usual. Additionally, I use a bit of colour-coding in this document (though it is not detrimental to anyone's understanding even if it weren't coloured). Words in purple are me talking to myself and other additional notes. Words in red are points of confusion that I should clear up soon.

Okay, enjoy!

1 Flows, Lie Derivatives, Frobenius

1.1 Flows

Let $p \in M$ and let $X \in \mathfrak{X}(M)$ be a smooth vector field.

Integral Curve $\gamma : (a, b) \subset \mathbb{R} \to M$ is an *integral curve* of X starting at p if $\gamma'(t) = X_{\gamma(t)}$ and $\gamma(0) = p$. We call it a *maximal integral curve* if its domain cannot be extended.

Proposition 1.1 (Existence of **Local Flows**)**.** *Let U be a coordinate open set and $p \in U$, let $X \in \mathfrak{X}(M)$, then there exists $\varepsilon > 0$ and neighbourhood $W \subset U$ of p and a C^∞ map $F : (-\varepsilon, \varepsilon) \times W \to U$, called the* local flow *near p generated by the vector field X, such that*

$$\frac{\partial}{\partial t} F(t, q) = X_{F(t,q)} \quad \textit{and} \quad F(0, q) = q$$

This means that for each fixed $q \in W$, the map $t \mapsto F_t(q) := F(t, q)$, called the flow line *of the local flow, is an integral curve of X starting at q defined on $(-\varepsilon, \varepsilon)$.*

Global Flow If F is defined on $\mathbb{R} \times M$ then we call it a *global flow*.

Complete Vector Field A vector field that admits a global flow is called a *complete vector field*.

Theorem 1.2 (Fundamental Theorem of Flows)**.** *Let $X \in \mathfrak{X}(M)$. Then:*

1. *For any $p \in M$, there exists a unique maximal integral curve starting at p.*
2. *There exists a unique maximal flow generated by X.*

1.2 Don't Lie To Me

1.2.1 Lie Derivative of Vector Fields

Lie Derivative of Vector Fields Given two smooth vector fields $X, Y \in \mathfrak{X}(M)$ and a point $p \in M$, the *Lie derivative* of Y in the direction of X at p is

$$\mathcal{L}_X Y|_p = \lim_{t \to 0} \frac{F_{-t_{*,F_t(p)}}(Y_{F_t(P)}) - Y_p}{t}$$

The first term on top $F_{-t_{*,F_t(p)}}(Y_{F_t(P)}) : \mathbb{R} \to T_pM$ is a vector in T_pM for every t. This limit always exists, and also $\mathcal{L}_X Y \in \mathfrak{X}(M)$ is another smooth vector field.

Think of the picture of the "flow a tangent vector from Y on X and see if it matches up with the X tangent vector of the flow."

Let $H(t) = F_{-t_{*,F_t(p)}}(Y_{F_t(P)})$ where $H : \mathcal{D}^{(p)} \to T_pM$. Here $\mathcal{D}^{(p)}$ is the maximal interval in which the maximal integral curve starting at p is defined. We see that $H'(0) = \mathcal{L}_X Y|_p = 0$ by the classic definition of a limit. Note that this is an important proof technique when dealing with Lie derivatives, it's helpful to think of the complicated limit in terms of a derivative of a single-variable function.

Invariant Under Flow A vector field Y is *invariant under the flow* of X if $F_{t_{*,p}}(Y_p) = Y_{F_t(p)}$ for all $p \in M$ and $t \in \mathcal{D}^{(p)}$.

Now suppose $X, Y \in \mathfrak{X}(M)$ are complete. X has flow F with time variable t, and Y has flow G with time variable s.

Commuting Flows The flows of X and Y *commute* if $F_t \circ G_s = G_s \circ F_t$ for all $s, t \in \mathbb{R}$.

Theorem 1.3 (Properties of Lie Being 0)**.** *Let $X, Y \in \mathfrak{X}(M)$ be complete. The following are equivalent:*

1. $\mathcal{L}_X Y = 0$
2. *Y is invariant under the flow of X*
3. *F_t, the flow of X, sends integral curves of Y to integral curves of Y*
4. *the flows of X and Y commute*
5. $\mathcal{L}_Y X = 0$

1.2.2 Lie Derivative of Functions

Let $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$.

Lie Derivative of Functions For $f \in C^\infty(M)$, we define the *Lie derivative* to be

$$\mathcal{L}_X f(p) = \lim_{t \to 0} \frac{F_t^*(f)(p) - f(p)}{t} = X(f)|_p = \left.\frac{d}{dt}\right|_{t=0} f \circ F_t(p)$$

Some computations here is confusing. Let $\gamma(t) := F_t(p)$ be the flow of X starting at p, then we can write not only that $\left.\frac{d}{dt}\right|_{t=0} f \circ F_t(p) = (f \dot{\circ} \gamma)(0)$ where the dot notation represents the usual derivative, but on the other hand we also have $\left.\frac{d}{dt}\right|_{t=0} f \circ F_t(p) = f_{*,F_0(p)} \circ \gamma_{*,0} = (f \circ \gamma)_{*,0}$. Then, as velocity vectors or tangent vectors in T_0M, we get that

$$(f \dot{\circ} \gamma)(0) \left.\frac{d}{dt}\right|_{t=0} = (f \circ \gamma)_{*,0} \left.\frac{d}{dt}\right|_{t=0}$$

We feed the identity function to both sides to get

$$\begin{aligned}\overbrace{(f \dot{\circ} \gamma)(0)}^{\text{number}} \underbrace{\left.\frac{d}{dt}\right|_{t=0}(\mathrm{id})}_{\text{vector eats id, gives 1}} &= f_{*,p} \circ \overbrace{\gamma_{*,0}\left(\left.\frac{d}{dt}\right|_{t=0}\right)}^{\text{def of flow}} \underbrace{(\mathrm{id})}_{\text{coord vector wrt chart}} \\ &= f_{*,0}(X_p)(\mathrm{id}) = X_p(\mathrm{id} \circ f) \\ &= X_p(f) = X(f)|_p\end{aligned}$$

1.2.3 Lie Bracket

The big theorem here is this:

Theorem 1.4 (Lie Derivative is Lie Bracket)**.**

$$\mathcal{L}_X Y = [X, Y]$$

Lie Bracket Let $X, Y \in \mathfrak{X}(M)$, their *Lie bracket* is defined to be

$$[X, Y] = XY - YX \in \mathfrak{X}(M)$$

Commuting Vector Fields We say X and $Y \in \mathfrak{X}(M)$ *commute* if $[X, Y] = 0$. This means that for all $f \in C^\infty(M)$
$$XY(f) = YX(f)$$

Proposition 1.5 (Properties of Lie Bracket). *The Lie bracket has the following properties:*

1. $\mathbb{R}$*-bilinearity*
2. $[X, Y] = -[Y, X]$
3. *Jacobi identity:*
$$\sum_{\text{cyclical}} [X, [Y, Z]] = 0$$
or, more specifically,
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Now we define some algebra stuff that we didn't learn in third year abstract algebra.

Lie Algebra A *Lie algebra over* $\mathbb{R}$ is a vector space V over $\mathbb{R}$ together with a product $[\cdot, \cdot] : V \times V \to V$, called the bracket, which satisfies the three properties above.

Algebra An *algebra over* $\mathbb{R}$ is a vector space V over $\mathbb{R}$ with a product $\cdot : V \times V \to V$ making V a ring satisfying the homogeneity condition, $a(X \cdot Y) = aX \cdot Y = X \cdot aY$.

A Lie algebra is not necessarily a ring because the bracket doesn't have to be associative. However, any algebra admits a Lie bracket (defined above, but using whatever $\cdot$ was in the generic algebra) which makes it a Lie algebra.

Derivation A *derivation* on a Lie algebra V is a map $D : V \to V$ that is $\mathbb{R}$-linear with respect to the vector space structure and satisfies the Leibniz rule with respect to the bracket: $D[X, Y] = [DX, Y] + [X, DY]$. Indeed, for $X \in \mathfrak{X}(M)$, $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a derivation on the Lie algebra $\mathfrak{X}(M)$.

Proposition 1.6 (Properties of Lie Derivative). *The Lie derivative satisfies the following properties:*

1. $\mathcal{L}_X Y = -\mathcal{L}_Y X$
2. $\mathcal{L}_X [Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z$ *since* $\mathcal{L}_X$ *is a derivation on* $\mathfrak{X}(M)$
3. $\mathcal{L}_{[X,Y]} Z = \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z$
4. *Let* $g \in C^\infty(M)$*, then* $\mathcal{L}_X(gY) = (\mathcal{L}_X g)Y + g\mathcal{L}_X Y$

Claim 1.7 (Coordinate Vector Fields Commute). *Let* $(U, \phi = (x^1, ..., x^n))$ *be a chart, then by Clairaut's theorem, for all* $f \in C^\infty(U)$*, we have*
$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$
Hence, $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$*, so coordinate vector fields commute.*

k-Frame If the k smooth vector fields $X_1, .., X_k \in \mathfrak{X}(M)$ are such that $X_{1p}, ..., X_{kp}$ are linearly independent for all $p \in M$, then we call them a *k-frame*.

1.3 Frobenius

Distribution For $p \in M$, choose a k-dimensional subspace $\Delta p \subset T_pM$. A *rank-k distribution* is $\Delta = \cup \Delta p \subset TM$. Basically, at each p we have a k-dim subspace of the tangent space, and we just put these all together and it's like a choice of k vector fields, kind of.

Smooth Distribution A rank-k distribution Δ is *smooth* if for all $p \in M$, there is a neighbourhood $U \subset M$ of p and $X_1, ..., X_k \in \mathfrak{X}(U)$ such that $\Delta q = \text{span}\{X_{1q}, ..., X_{kq}\}$, for all $q \in U$.

Local Frame In the above case, say $X_1, ..., X_k \in \mathfrak{X}(U)$ is a *local frame* of Δ near p. Treat it as a basis of the chosen k-dim subspace at p.

Section A *section* of Δ is a map $X : M \to \Delta \subset TM$ such that $\pi \circ X = \text{id}$. If Δ is smooth, then denote the space of smooth sections of Δ by $\Gamma(\Delta)$, and $\Gamma(\Delta) \subset \Gamma(M) = \mathfrak{X}(M)$.

Integral Submanifold Let Δ be a smooth rank k distribution. For $p \in M$, if there exists a submanifold S containing p with the property that $T_qS = \Delta_q$ for all $q \in S$, then we call S an *integral submanifold* of Δ. This means that each of the k vector fields (chosen in the distribution) in Δ is tangent to S, some k-dim submanifold.

Integrable A smooth rank k distribution Δ is *integrable* if for all $p \in M$, there exists an integral submanifold of Δ containing p.

Involutive A smooth rank k distribution Δ is *involutive* if for every local frame $X_1, ..., X_k$ of Δ, we have that $[X_i, X_j] \in \Gamma(\Delta)$, i.e.,

$$[X_i, X_j] = \sum_{\ell=1}^{k} c_{ij}^{\ell} X_\ell \text{ for } c_{ij}^{l} \in C^\infty(U)$$

This is equivalent to saying that $\Gamma(\Delta)$ is a Lie subalgebra of $\Gamma(M)$.

Theorem 1.8 (Frobenius Theorem)**.** *A smooth rank-k distribution Δ is integrable iff it is involutive.*

Flat Chart We say that a chart (U, ϕ) is *flat* for Δ if for any $q \in U$, with $\phi(q) = (q^1, ..., q^n)$, the set $S_q = \{x^{k+1} = q^{k+1} = \cdots = x^n = q^n = 0\}$ defined by the vanishing of the last $n-k$ coordinates is an integral submanifold of Δ containing p.

Completely Integrable Δ is *completely integrable* if there exists a flat chart near every point.

Theorem 1.9 (Strengthed Frobenius Theorem)**.** *Given a smooth rank k distribution Δ, the following are equivalent:*

- Δ *is completely integrable.*
- Δ *is integrable.*
- Δ *is involutive.*

2 Covector Fields, Differential 1-forms

Let M be a manifold and let $p \in M$.

2.1 Dual Vectors

You thought tangent vectors were confusing? Well, we were just beginning! Now we will define a whole new world of things called dual vectors on top of the already-confusing tangent vectors. Haha! #getrekt.

Dual Vectors, Covectors $T_p^*M := \{f : T_pM \to \mathbb{R} \,|\, f \text{ is linear}\}$.

Dual Basis Let $\{v_1, ..., v_n\}$ be a basis of T_pM. Define $\theta^i \in T_p^*M$ by $\theta^i(v_j) = \delta^i_j$. Then, $\{\theta^1, ..., \theta^n\}$ is a basis for T_p^*M and it is called the *dual basis* of $\{v_1, ..., v_n\}$.

Covector Field We define a *covector field* as a choice of a covector in T_p^*M at each point p. This is a map $p \mapsto \theta \in T_p^*M$ which eats vector fields and gives functions.

Cotangent Bundle The *cotangent bundle* is:

$$\begin{aligned} T^*M &:= \cup_{p\in M} T_p^*M \\ &= \{(p, \theta) \,|\, p \in M,\, \theta \in T_p^*M\} \end{aligned}$$

We have the natural map $\pi : T^*M \to M$ defined by $(p, \theta_p) \mapsto p$ where $\theta_p \in T_p^*M$.

Topology on T^*M Lazy. Refer to lecture notes. It's roughly the same idea as when we did it for TM.

2.2 Differential 1-forms

1-form A *section* ω of T^*M as a map $\omega : M \to T^*M$ satisfying $\pi \circ \omega : \mathrm{id}_\omega$. This is, like, a covector field. This is also known as a *differential 1-form*.

Smooth 1-form As you might have guessed already, a *smooth 1-form* ω is a C^∞ section of T^*M.

Differential For $f \in C^\infty(M)$, we defined the 1-form $df : M \to T^*M$ where $df_p \in T_p^*M$ is defined for $v \in T_pM$ by

$$df_p(v) = v(f) = f_{*,p}(v)(\mathrm{id})$$

Proposition 2.1 (differential of f at p)**.** *The following diagram commutes:*

$$\begin{tikzcd} T_pM \arrow[r, "df_p"] \arrow[dr, "f_{*,p}"'] & \mathbb{R} \arrow[d, "\phi"] \\ & T_{f(p)}\mathbb{R} \end{tikzcd}$$

where $\phi(c) = c\frac{d}{dx}|_{x=f(p}$. *Under the identification* $T_{f(p)}\mathbb{R} \cong \mathbb{R}$, *we have* $f_{*,p}$ "$\cong$"df_p. *Both are called the* differential *of* f *at* p.

Coordinate Dual Basis Let $(U, \phi = (x^1, ..., x^n))$ be a coordinate chart near p. Then $x^i \in C^\infty(U)$, and so dx^i is a 1-form satisfying

$$dx^i_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \left. \frac{\partial}{\partial x^i} \right|_p (x^i) = \delta^i_j$$

So $\{dx^1_p, ..., dx^n_p\}$ is the dual basis of $\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, ..., \left. \frac{\partial}{\partial x^n} \right|_p \right\}$, and we call it the *coordinate dual basis*.

Independence of Coordinate Chart For any $f \in C^\infty(U)$, we have $df_p = a^i dx^i_p$ for $a^i \in \mathbb{R}$. Applying $\frac{\partial}{\partial x^i}\Big|_p$ to both sides, we get:

$$\frac{\partial f}{\partial x^i}\Big|_p = df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = a^i dx^i_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = a^j$$

Hence, we know $df = \frac{\partial}{\partial x^i}dx^i$ on U. In a different coordinate chart, $(U, \psi = (y^1, ..., y^n))$, we get $df = \frac{\partial}{\partial y^i}dy^i$. Note that

$$\begin{aligned}\frac{\partial f}{\partial y^i}dy^i &= \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial y^i}\frac{\partial y^i}{\partial x^k}dx^k \\ &= \frac{\partial f}{\partial x^j}\delta^j_k dx^k \\ &= \frac{\partial f}{\partial x^i}dx^i\end{aligned}$$

2.3 Smoothness Criteria for 1-forms

Now we present two more criteria for the smoothness of 1-forms.

Action of 1-forms on $\mathfrak{X}(M)$ A 1-form ω acts on $\mathfrak{X}(M)$; it eats $X \in \mathfrak{X}$ and spits out $\omega(X) \in C(M)$, a real-valued function on M. For $X \in \mathfrak{X}(M)$, define $\omega(X) : M \to \mathbb{R}$ by $p \mapsto \omega_p(X_p)$. This function eats a point p, uses X to choose a tangent vector X_p, and then applies ω to it. This action is C^∞-linear. What's even better is the following theorem:

Theorem 2.2. *Let $A : \mathfrak{X}(M) \to C^\infty(M)$ be a C^∞-linear map. Then there exists a unique smooth 1-form ω such that the action of ω on $\mathfrak{X}(M)$ coincides with A.*

Lemma 2.3. *$A(X)(p)$ only depends on X_p, i.e., $A(X)(p) = A(Y)(p) \iff A(X - Y)(p) = 0$*

Proposition 2.4 (C^∞ Criterion for 1-forms)**.** *Let $\omega : M \to T^*M$ be a 1-form. The following are equivalent:*

1. *ω is smooth as a section over T^*M.*
2. *On any chart (U, ϕ), we have $\omega = a_i dx^i$ where $a_i \in C^\infty(U)$.*
   In particular, dx^i are smooth 1-forms on U.
3. *By its action on $\mathfrak{X}(M)$, $\omega(X) \in C^\infty(M)$ whenever $X \in \mathfrak{X}(M)$.*
   This says: any smooth 1-form ω defines a C^∞-linear map $\omega : \mathfrak{X}(M) \to C^\infty(M)$

Space of Smooth 1-forms We denote the *space of all smooth 1-forms* by $\Omega^1(M)$. It is simultaneously the space of all smooth sections of T^*M and the space of all $\omega : \mathfrak{X}(M) \to C^\infty(M)$ such that ω is C^∞-linear. It is a module over $C^\infty(M)$, and a vector space over $\mathbb{R}$. In fact:

Proposition 2.5 (Module Isomorphism of $\Omega^1(M)$)**.** *Now that we have the three equivalent definitions, we have that $\Phi : \left\{\text{smooth sections } \omega : M \to T^*M\right\} \to \left\{\omega : \mathfrak{X} \to C^\infty(M) \,\middle|\, \omega \text{ is } C^\infty\text{-linear}\right\}$ defined by*

$$\begin{aligned}\omega : M \to T^*M \mapsto \omega : \mathfrak{X}(M) &\to C^\infty(M) \\ X &\mapsto \big(\omega(X) : p \mapsto \omega_p(X_p)\big)\end{aligned}$$

is an isomorphism of modules.

2.4 Pullback of 1-forms

Let $F : N \to M$ be a C^∞ map. Recall that $F_{*,p} : T_pN \to T_{F(p)}M$ is the pushforward of F at p.

$$\begin{tikzcd} T_pN \arrow[r, "F_{*,p}"] \arrow[dr, bend right, "(\theta\circ F_{*,p})\in T_p^*N"'] & T_{F(p)}M \arrow[d, "\theta\in T^*_{F(p)}M"] \\ & \mathbb{R} \end{tikzcd}$$

Pullback of a covector Given a dual vector $\theta \in T^*_{F(p)}M$, the *pullback* by F is defined to be

$$\begin{aligned} F^{*,p} : T^*_{F(p)}M &\to T^*_pN \\ \theta &\mapsto \theta \circ F_{*,p} \end{aligned}$$

Even though vector fields cannot always be pushed forward, the fun thing about 1-forms is that they can always be pulled back.

Pullback of a 1-form Let ω be a 1-form on M, then the *pullback* of ω is a 1-form on N defined by

$$\begin{aligned} F^*\omega : N &\to T^*N \\ p &\mapsto (F^*\omega)_p = F^{*,p}(\omega_{F(p)}) = \omega_{F(p)} \circ F_{*,p} \end{aligned}$$

By its action as a 1-form on $\mathfrak{X}(N)$, $F^*\omega$ satisfies, for $X \in \mathfrak{X}(M)$,

$$\begin{aligned} F^*\omega(X)(p) &= (F^*\omega)_p(X_p) \\ &= \omega_{F(p)} \circ F_{*,p}(X_p) \end{aligned}$$

If F is a diffeomorphism, then $F^*\omega(X) = \omega(F_*(X))$. So we can write $F^*\omega = \omega \circ F_*$.

Proposition 2.6 (Properties of Pullback)**.** *These are the properties of the pullback of a 1-form:*

1. $F^*(g\omega) = F^*(g)F^*(\omega) = (g \circ F)F^*(\omega)$.
2. *if* $g \in C^\infty(M)$*, then* $F^*(dg) = d(F^*g) = d(g \circ F)$.
3. *if* $\omega \in \Omega^1(M)$*, then* $\omega \in \Omega^1(N)$ *and* $F^* : \Omega^1(M) \to \Omega^1(N)$ *and is* $\mathbb{R}$*-linear with respect to the vector space structure of* $\Omega^1(M)$ *and* $\Omega^1(N)$.

I give an example for the k-form case later, on page 13.

3 Tensors Make Me Tense, Differential Forms Make Me Tenser

Maybe if I keep learning about tensors and forms, by the 20th time around I'll finally be able to wrap my head around them?

3.1 Alternating Tensors

Let V be a vector space. This is all so technical, I hate it.

k-Tensor A *k-tensor* is a map $T : \overbrace{V \times \cdots \times V}^{k \text{ times}} \to \mathbb{R}$ that is multilinear, i.e. linear in each component.

Signed k-Dimensoinal Volume Meter We are interesed in tensors that give us a notion of a *signed k-dimensoinal volume meter*, which means that it satisfies the following:

1. $f : V \times \cdots \times V \to \mathbb{R}$ is defined by $(v_1, ..., v_k) \mapsto$ the signed k-dim volume of parallelepiped $P_{v_1,...,v_k}$.
2. f is multilinear
3. f is alternating, i.e. $f(v_1, ...v_i, ..., v_j, ..., v_k) = -f(v_1, ..., v_j, ...v_i, ..., v_k)$.

Alternating k-Tensor An *alternating k-tensor* or a *k-covector* is a k-tensor on V that is alternating.

Proposition 3.1 (Properties of Alternating k-Tensor)**.** *Let f be a k-tensor. The following are equivalent:*

1. *f is alternating.*
2. *$f(v_1, ..., v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$.*
3. *$f(v_1, ..., v_k) = 0$ whenever $v_1, ..., v_k$ are linearly dependent.*
4. *for any $\sigma \in S_k$, $\sigma f(v_1, ..., v_k) = f(v_{\sigma(1)}, ..., v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, ..., v_k)$.*

Example 3.1. Let $\theta^1, \theta^2 \in V^*$ be dual vectors. Then the wedge product of dual vectors, $\theta^1 \wedge \theta^2 : (v_1, v_2) \mapsto \theta^1(v_1)\theta^2(v_2) - \theta^1(v_2)\theta^2(v_1)$, is an alternating 2-tensor.

Denote by $T_k(V)$ the vector space of k-tensors.
Denote by $A_k(V)$ the vector space of alternating k-tensors.
Denote by $S_k(V)$ the vector space of symmetric k-tensors.

Sym Operator Define the projection operator $\operatorname{Sym} : T_k(V) \to S_k(V)$ by

$$f \mapsto \frac{1}{k!} \sum_{\sigma \in S^k} \sigma f$$

Then $\operatorname{Sym}(f) = f$ iff f is symmetric.

Alt Operator Similarly, define $\operatorname{Alt} : T_k(V) \to A_k(V)$ by

$$f \mapsto \frac{1}{k!} \sum_{\sigma \in S^k} \operatorname{sgn}(\sigma) \sigma f$$

Then $\operatorname{Alt}(f) = f$ iff f is alternating.

Tensor Product The *tensor product* $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$ for $f \in T_k(V)$, $g \in T_l(V)$ by

$$f \otimes g(v_1, ..., v_k, w_1, ..., w_l) = f(v_1, ..., v_k) g(w_1, ..., w_l)$$

The tensor product is associative, i.e. $f \otimes (g \otimes h) = (f \otimes g) \otimes h$, so we can just write $f \otimes g \otimes h$.

3.2 Wedge Product

Wedge Product Define the *wedge product* $\wedge : A_k(V) \times A_l(V) \to A_{k+l}(V)$ by

$$f \wedge g = \frac{(k+1)!}{k!l!}\text{Alt}(f \otimes g) = \frac{1}{k!l!}\sum_{\sigma \in S_{k+l}} (\text{sgn}\sigma)\sigma(f \otimes g)$$

Note that when $f \in A_0(V) = \mathbb{R}$, we have $f \wedge g := fg$

Multi-index Read the definition and properties of the *multi-index notaton* by clicking on this paragraph and going to the hyperlinked Wikipedia page.

Proposition 3.2 (Properties)**.** *These are the properties of* $\wedge : A_k(V) \times A_l(V) \to A_{k+l}(V)$

1. *Bilinearity*
2. *Let* $\{e_1, ..., e_m\}$ *be a basis for* V*, and* $\{\alpha_1, ..., \alpha_n\}$ *be the dual basis of* $V^* = T_1(V) = A_1(V)$*. For any indexing set* $I = \{i_1, ..., i_k\} \subset \{1, ..., n\}$ *such that* $1 \leq i_1 < \cdots < i_k \leq n$*, we have the unique alternating* k*-tensor* α^I *satisfying*
$$\alpha^I(e_J) = \delta^I_J$$
for any multi-index J *in ascending order. Then,* $\{\alpha^I : \text{multi-index in ascending order}\}$ *makes a basis for* $A_k(U)$*, and so* $\dim(A_k(U)) = \binom{n}{k}$*.*
3. $\alpha^I \wedge \alpha^J = \alpha^{IJ}$
4. *Associativity:* $(f \wedge g) \wedge h = f \wedge (g \wedge h)$
5. *Anti-commutativity:* $f \wedge g = (-1)^{lk} g \wedge f$
6. *For any* $\theta^1, ..., \theta^k \in V^*$ *and any* $v_1, .., v_k \in V$*,*
$$\theta^1 \wedge \cdots \wedge \theta^k(v_1, ..., v_k) = \det(\theta^i(v_j))$$

In fact, the wedge product is the unique bilinear, associative, and anti-commutative product $A_k(V) \times A_l(V) \to A_{k+l}(V)$ *satisfying* $\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$*.*

Alternating Tensors With the new understanding from tensor products, denote the vector space of alternating k-tensors $A_k(V)$ by $\Lambda^k(V^*) = \text{span}\{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} | 1 \leq i_1 < \cdots < i_k \leq n\}$, and define

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*)$$

Claim 3.3 (Algebraic Property of Lambda)**.** *The wedge product makes* $\Lambda(V^*)$ *an associative anticommutative graded algebra of dimension* $2n$*. See definitions below.*

Graded Algebra An algebra A over $\mathbb{R}$ is *graded* if it can be written as a direct sum of vector spaces A^k over $\mathbb{R}$,

$$A = \bigoplus_{k=0}^{\infty} A^k$$

such that the multiplication map sends $A^k \times A^l \to A^{k+l}$.

Anticommutativity A graded algebra is *anticommutative* if t satisfies $ab = (-1)^{lk} ba$ for all $a \in A^k$, $b \in A^l$.

3.3 Intro to Differential Forms

Let M be a smooth manifold, $p \in M$, and (U, ϕ) a chart near p. Then:

- $\left\{ \frac{\partial}{\partial x^1}\Big|_p, ..., \frac{\partial}{\partial x^n}\Big|_p \right\}$ is the coordinate basis for T_pM
- $\{dx_p^1, ..., dx_p^n\}$ is the coordinate dual basis
- $\{dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k} \mid 1 \leq i_1 < \cdots i_k \leq n\}$ is a basis for $\Lambda^k(T_p^*M)$.

Bundle of Alternating k-Tensors The *bundle of alternating k-tensors* is $\Lambda^k(T^*M) := \bigcup_{p \in M} \Lambda^k(T_p^*M)$, which comes with the map $\pi : \Lambda^k(T^*M) \to M$ defined by $(p, \omega_p) \mapsto p$ where $\omega_p \in \Lambda^k(T_p^*M)$.

Section, k-Forms A *section* of $\Lambda^k(T^*M)$ is a map $\omega : M \to \Lambda^k(T^*M)$, where $p \mapsto (p, \omega_p)$ with $\omega_p \in \Lambda^k(T_p^*M)$, such that $\pi \circ \omega = \mathrm{id}_M$. Sections of $\Lambda^k(T^*M)$ are called *differential k-forms.*

FORMS PICK OUT TENSORS AT POINTS! You feed a point to a form, and it goes "hmm *this* one" and gives you a tensor at that point.

Wedge Product Extended Let ω be a k-form, ν be an l-form on M, we can defined the *wedge product* which is a $k + l$ form:

$$\begin{aligned} \omega \wedge \nu : M &\to \Lambda^{k+l}(T^*M) \\ p &\mapsto (p, (\omega \wedge \nu)_p := \omega_p \wedge \nu_p \end{aligned}$$

So, since $dx_p^I = dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k}$, for all $p \in U$, then $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

Action of k-Forms On $\mathfrak{X}(M)$ We can define the action of k-forms on $\mathfrak{X}(M)$. For $X_1, .., X_k \in \mathfrak{X}(M)$,

$$\begin{aligned} \omega(X_1, ..., X_k) : M &\to \mathbb{R} \\ p &\mapsto \omega_p(X_{1p}, ..., X_{kp}) \end{aligned}$$

This action $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C(M)$ (where $C(M)$ is the space of real-valued functions on M) is C^∞-linear.

Smoothness Criteria for k-Forms Let ω be a 1-form. The following are equivalent:

1. ω is smooth as a section
2. on any chart, $\omega = c_I dx^I$ on U where $c_I \in C^\infty(U)$
3. the action on $\mathfrak{X}$ sends vector fields to a smooth function, i.e. $\omega(X_1, ..., X_k) \in C^\infty(M)$ whenever $X_1, ..., X_k \in \mathfrak{X}(M)$.

Thus, $\Phi : \left\{\text{smooth sections } \omega : M \to \Lambda^k(T^*M)\right\} \to \left\{\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M) \text{ where } \omega \text{ is } C^\infty\text{-multilinear and alternating}\right\}$ defined by $(\omega : M \to \Lambda^k(T^*M)) \mapsto (\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M))$ is an isomorphism of modules over $C^\infty(M)$.

Vector Space of k-Forms We can define the vector space of C^∞ differential forms on n-dimensional manifold M to be the direct sum

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$$

With the wedge product, $\Omega^*(M)$ is an associative anticommutative graded algebra.

3.3.1 Pullback of k-Forms

Today is August 18, the day before the final, and I have only learned how to do this today thanks to thomas. Let's see if I can put it into words.

Let $F : N \to M$ be a C^∞ map.

Pullback of k-Covector We defined the *pullback* of k-covectors by F as

$$\begin{aligned} F^{*,p} &: \Lambda(T^*_{F(p)}M) \to \Lambda(T^*_p N) \\ &\theta \mapsto F^{*,p}(\theta) : (v_1, ..., v_k) \mapsto \theta(F_{*,p}v_1, ..., F_{*,p}v_k) \end{aligned}$$

which has the property that for $\theta_1 \in \Lambda^k(T^*_{F(p)}M)$ and $\theta_2 \in \Lambda^l(T^*_{F(p)}M)$,

$$F^{*,p}(\theta_1 \wedge \theta_2) = F^{*,p}(\theta_1) \wedge F^{*,p}(\theta_2) \in \Lambda^{k+l}(T^*_{F(p)}M)$$

Pullback of k-Form For $\omega \in \Omega^k(M)$, define the pullback by F by

$$\begin{aligned} F^*\omega : N &\to \Lambda^k(T^*N) \\ p &\mapsto (p, F^{*,p}(\omega_{F(p)})) \end{aligned}$$

who has the property that for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

and we can write this in coordinates as

$$\begin{aligned} F^*(\omega) &= F^*(aIdx^I) \\ &= (a_I \circ F)d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k}) \end{aligned}$$

which means that $F^*\omega \in \Omega^k(N)$ is now a k-form on N, so $F^* : \Omega^k(M) \to \Omega^k(N)$.

Proposition 3.4 (Properties of Pullback)**.** *To summarize, we have:*

1. F^* *is* $\mathbb{R}$*-linear*
2. $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$
3. *if* $\omega \in \Omega^k(M)$*, then* $F^*\omega \in \Omega^k(N)$

Example 3.2. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $F(x, y) = (xy, e^x) = (u, v)$ and $\omega = xdy \in \Omega^1(\mathbb{R}^2)$, verify that $F^*d\omega = dF^*\omega$. We can compute the following things:

- $d\omega = dx \wedge dy$
- $DF = \begin{pmatrix} y & x \\ e^x & 0 \end{pmatrix}$, which means $dF^1 = ydx + xdy$ for the first component function F^1, and $dF^2 = e^x dx$.
- Since $u \circ F = xy$ and $v \circ F = e^x$, we have

$$\begin{aligned} F^*\omega &= (F^*u)(d(F^*v)) \\ &= (u \circ F)d(v \circ F) \\ &= xye^x dx \end{aligned}$$

- $dF^*\omega = xe^x dy \wedge dx$
- $F^*(d\omega) = dF^1 \wedge dF^2 = (xdy) \wedge (e^x dx) = xe^x dy \wedge dx$

4 Cartan Calculus

4.1 Exterior Derivative

The exterior derivative is "defined" by the following properties, which would allow for Stokes' Theorem.

Theorem 4.1 (Exterior Derivative Desired Properties)**.** *There exists a collection of $\mathbb{R}$-linear maps, called the* exterior derivative*, $d : \Omega^k(M) \to \Omega^{k+1}(M)$, which satisfies:*

1. *For $f \in \Omega^0(M)$, df is the differential of f.*
2. *For $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$, we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-)^k \omega \wedge d\eta$.*
3. *$d \circ d = d^2 = 0$*

Antiderivation On a graded algebra $A = \bigoplus_{k=1}^{\infty} A^k$, an *antiderivation* is an $\mathbb{R}$-linear map $D : A \to A$ satisfying

$$D(\omega\tau) = D(\omega)\tau + (-1)^k \omega D\tau$$

for $\omega \in A^k$ and $\tau \in A^l$. The antiderivation is of *degree m* if $\deg(D\omega) = \deg(\omega) = m$, for all $\omega \in A^k$.

Exterior Derivative Finally, we define it. Let $\omega \in \Omega^k(M)$. On a chart (U, ϕ), where $\omega = a_I dx^I$ such that $I = \{i_1, ..., i_n\}$, we define

$$d\omega = da_I \wedge dx^I$$

on U. Recall that

$$da_I \wedge dx^I = \sum_{1 \le i_1 < .. < i_k \le n} da_{(i_1, ..., i_n)} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_n}$$

If $k = 0$, then yay, d is the differential on $\Omega^0(M)$ as defined before, where on each chart we have

$$df = \frac{\partial f}{\partial x^i} \wedge dx^i$$

To be honest, this formula means nothing to me, and that's probably a bad thing. However, I do know how to work with it, so the way it works is like this example:

Example 4.1. On $\mathbb{R}^3$, let $\beta = y dx \wedge dz$, then

$$\begin{aligned} d\beta &= \frac{\partial}{\partial x}(y dx \wedge dz) \wedge dx + \frac{\partial}{\partial y}(y dx \wedge dz) \wedge dy + \frac{\partial}{\partial z}(y dx \wedge dz) \wedge dz \\ &= dx \wedge dz \wedge dy \\ &= -dx \wedge dy \wedge dz \end{aligned}$$

Corollary (Exterior Derivative Properties)**.** *There exists a collection of $\mathbb{R}$-linear maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying:*

1. *$d : \Omega^0(M) \to \Omega^1(M)$ is the differntial.*
2. *d is an antiderivation of degree 1.*
3. *$d^2 = 0$.*
4. *(A special bonus one!) Let $F : N \to M$ be a smooth map. Then on $\Omega^k(M)$,*

$$F^* \circ d = d \circ F^*$$

Theorem 4.2 (Global Intrinsic Formula For The Exterior Derivative)**.** *Let $\omega \in \Omega^k(M)$. Then for $X_0, ..., X_k \in \mathfrak{X}(M)$,*

$$d\omega(X_0, ..., X_k) = \sum_{i=0}^{k} (-1)^{i-1} X_i(\omega(X_0, ..., \hat{X}_i, ..., X_k)) - \sum_{0 \le i \le j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k)$$

4.2 Interior Multiplication

4.2.1 On a Vector Space

Let $\beta \in \Lambda^k(V^*)$ for $k \geq 2$, let $v \in V$.

Interior Multiplication / Contraction Define $\iota_v(\beta) \in \Lambda^{k-1}(V^*)$ by

$$\iota_v(\beta)(v_1, ..., v_{k-1}) = \beta(v, v_1, ..., v_{k-1})$$

for $v_1, ..., v_{k-1} \in V$. This is called the *interior multiplication* or *contraction* of β with V. Note:

- If $k = 0$, then $\iota_v(\beta) := 0$
- If $k = 1$, then $\iota_v(\beta) := \beta(v)$

Proposition 4.3 (Properties of Interior Multiplication on a Vector Space)**.** *Fix $v \in V$.*

1. *If $\alpha_1, ..., \alpha_k \in V^* = \Lambda^1(V^*)$, then*

$$\iota_v(\alpha_1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \hat{\alpha^i} \wedge \cdots \wedge \alpha^k$$

2. $\iota_v \circ \iota_v = 0$
3. *given $\beta \in \Lambda^k(V^*)$ and $\gamma \in \Lambda^l(V^*)$,*

$$\iota_v(\beta \wedge \gamma) = \iota_v(\beta) \wedge \gamma + (-1)^k \beta \wedge \iota_v(\gamma)$$

So, $\iota_v : \Lambda^*(V^*) \to \Lambda^*(V^*)$ is an antiderivation of degree (-1) on the graded algebra $\Lambda^*(V^*)$ whose square is 0.

4.2.2 On Manifolds

Fix $X \in \mathfrak{X}(M)$ and let $\omega \in \Omega^k(M)$.

Interior Multiplication of Forms Define $\iota_X \omega$ as the $k-1$-form given by

$$(\iota_X \omega)_p = \iota_{X_p} \omega_p$$

For $X_1, ..., X_{k-1} \in \mathfrak{X}(M)$, $\iota_X \omega(X, ..., X_{k-1}) = \omega(X, X_1, ..., X_{k-1}) \in C^\infty(M)$.
If $k = 0$, then $\iota_X \omega = 0$.
If $k = 1$, then $\iota_X \omega = \omega(X)$.

Proposition 4.4 (Properties of Interior Multiplication)**.** *$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ satifies*

1. *It is $\mathbb{R}$-linear.*
2. *$\iota(\omega \wedge \eta) = \iota_X(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_X(\eta)$, for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$.*
3. *$\iota_X \circ \iota_X = {\iota_X}^2 = 0$*

So, ι_X is an antiderivation on the graded algebra $\Omega^*(M)$ of degree (-1) such that $\iota_X^2 = 0$.

Also,

$$\iota : \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^{k-1}(M)$$

is C^∞-bilinear. This means that $(\iota_X \omega)_p$ depends only on X_p and p.

Proposition 4.5. *For $X, Y \in \mathfrak{X}(M)$,*

$$\iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0$$

4.3 Lie Derivative on Forms

Lie Derivative Given a k-form $\omega \in \Omega^k(M)$, the *Lie derivative* $\mathcal{L}_X\omega \in \Omega^k(M)$ is a k-form defined pointwise as follows:

$$\mathcal{L}_X\omega|_p = \lim_{t\to p} \frac{F_t^*(\omega_{F_t(p)}) - \omega_p}{t} = \frac{d}{dt}\bigg|_{t=0} \underbrace{F_t^*(\omega_{F_t(p)})}_{\text{curve in } \Lambda^k(T_p^*M)}$$

We can feed it a tangent vector v at p. For $v \in T_pM$,

$$\mathcal{L}_X\omega(v) = \frac{d}{dt}\bigg|_{t=0} \omega_{F_t(p)}(F_{t*}v)$$

Theorem 4.6 (Properties of Lie Derivative)**.** *Let* $X \in \mathfrak{X}(M)$.

1. $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ *is a derivation (* $\mathbb{R}$*-linear and satisfies* $\mathcal{L}_X(\omega\wedge\eta) = \mathcal{L}_X\omega\wedge\eta + \omega\wedge\mathcal{L}_X\eta$*).*
2. $\mathcal{L}_X d = d\mathcal{L}_X$.
3. $\mathcal{L}_X(\omega(X_1, ..., X_k)) = \mathcal{L}_X\omega(X_1, ..., X_k) + \sum_{i=1}^k \omega(X_1, ..., \mathcal{L}_X X_i, ..., X_k)$.
4. ***Cartan's Magic Formula.*** $\mathcal{L}_X = \iota_X d + d\iota_X$.
5. $\mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X = \iota_{[X,Y]}$.
6. $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$

Proposition 4.7. *Let* $\omega \in \Omega^1(M)$*, let* $X, Y \in \mathfrak{X}(M)$*. Then*

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Proposition 4.8. *Given a 1-form* $f \in \Omega^0(M)$*, we have* $\mathcal{L}_X f = X(f)$.

4.4 Summary!

We introduced 3 opertaors on $\Omega^*(M)$:

$$\begin{aligned} d &: \Omega^k(M) \to \Omega^{k+1}(M) && \text{antiderivation of degree 1} \\ \iota_X &: \Omega^k(M) \to \Omega^{k-1}(M) && \text{antiderivation of degree } -1 \\ \mathcal{L}_X &: \Omega^k(M) \to \Omega^k(M) && \text{derivation} \end{aligned}$$

And we have this big **list of facts**. Good luck have fun:

- $d^2 = 0 = \iota_X^2$
- $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$
- $\iota_X\iota_Y + \iota_Y\iota_X = 0$
- $d\mathcal{L}_X - \mathcal{L}_X d = 0$
- $\mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X = \iota_{[X,Y]}$
- $d\iota_X + \iota_X d = \mathcal{L}_X$

5 Just Got Here? Welcome to Orientation!

5.1 Orientation of Vector Spaces

Let V be an n-dimensional vector space. Recall that an orientation on $\mathbb{R}^n$ can be specified by fixing an ordered basis.

Change of Basis Definition Let $\alpha = \{v_1, ..., v_n\}$, $\beta = \{w_1, ..., w_n\}$ be ordered bases of V. We can define an equivalence relation: α, β specify the same orientation if $A_\alpha^\beta = \begin{bmatrix}[v_1]_\beta & \cdots & [v_n]_\beta\end{bmatrix}$ has positive determinant. A_α^β satisfies:

1. $v_i = w_j A_{\alpha_i}^{\beta_j}$
2. $A_\alpha^\beta : [v]_\alpha \mapsto [v]_\beta$
3. $A_\alpha^\beta = (A_\beta^\alpha)^{-1}$

Each class is called an orientation on V. If we fix $\gamma \in \Lambda^n(V^*)$ and let α, β be two ordered bases for V, then

$$\gamma(v_1, ..., v_n) = \det(A_\beta^\alpha \gamma(\omega_1, ..., \omega_n)$$

5.2 Orientation of Manifolds

Pointwise Orientation A *pointwise orientation* on M is a choice of orientation on each T_pM. We have $2^{|M|}$ choices of pointwise orientation.

Orientation 1 An *orientation* on M is a pointwise orientation on M such that for all $p \in M$, there is a local frame $X_1, ..., X_n \in \mathfrak{X}(U)$ such that $\{X_{1q}, ..., X_{nq}\}$ is consistent with the orientation specified on T_qM, for all $q \in U$.

Orientation 2 An *orientation* on M is a pointwise orientation on M such that for all $p \in M$, there exists a chart (U, ϕ) near p such that $\left\{ \frac{\partial}{\partial x^1}\Big|_q, ..., \frac{\partial}{\partial x^n}\Big|_q \right\}$ is consistent with the orientation specified on T_qM, for all $q \in U$.

Oriented Atlas An atlas is called an *oriented atlas* if between any two charts (U, ϕ) and (V, ψ), the transition map satisfied $\det(D\psi \circ \phi^{-1}) > 0$ on $U \cap V$.

Orientation 3 An *orientation* on M is a pointwise orientation on M that admits an oriented atlas.

Equivalence On Oriented Atlases On the space of all oriented atlases, we can define the following equivalence relation: $\mathcal{A} \sim \mathcal{A}'$ specify the same orientation iff $\mathcal{A} \cup \mathcal{A}'$ us another oriented atlas. Each equivalence class represents an orientaton on M.

Orientation 4 An *orientation* on M is a pointwise orientation on M such that for all $p \in M$, there is a chart (U, ϕ) such that $dx_q^1 \wedge \cdots \wedge dx_q^n$ specifies the same orientation on T_qM for all $p \in U$.

5.3 Orientability

Orientable A manifold M is *orientable* if it admits an orientation. An *oriented manifold* is an orientable manifold that comes with an orientation.

Non-Examples The Möbius strip, the Klein bottle, and $\mathbb{R}P^n$ are all examples of *non*-orientable manifolds.

Proposition 5.1. *An orientable manifolds admits 2^C orientations, where C is the number of connected components in M.*

Theorem 5.2. *A manifold is orientable iff there exists a C^∞ nowhere vanishing n-form on M.*

6 Know Your Boundaries

Manifolds with boundary. This should be very similar to what we did at the beginning of the course, but instead of $\mathbb{R}^n$ we now base it on a space with a convenient boundary.

The prototype of a manifold with boundary is the upper half space

$$H^n = \{(x^1, ..., x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

with the subspace topology.

1. Points with $x^n > 0$ are called *interior points* of H^n, denoted by $(H^n)^\circ$
2. Points with $x^n = 0$ are called *boundary points* of H^n, denoted by ∂H^n.

Topological n-Manifold with Boundary A *topological n-manifold with boundary* is a second-countable and Hausdorff topological space that is locally H^n.

Smooth Manifold with Boundary A *smooth manifold with boundary* is a topological manifold with boundary together with a maximal atlas.

1. A point $p \in M$ is an *interior point* if there exists a chart (U, ϕ) near p such that $\phi(p) \in (H^n)^\circ$
2. A point $q \in M$ is a *boundary point* if there exists a chart (V, ψ) near q such that $\psi(q) \in \partial H^n$

Proposition 6.1. *The notions of interior and boundary points is well-defined and independent of coordinates. They partition M into $M^\circ = \{interior\ points\}$ and $\partial M = \{boundarypoints\}$.*

Things That Carry Over The things that carry over to a manifold with boundary as expected are: smooth maps, tangent vectors, T_p^*M, embedded / regular submanifolds.

Theorem 6.2. *If M is an n-dimensional manifold with boundary, then ∂M is an $(n-1)$-dimensional submanifold of M without boundary, with the abuse of notation $i_{*,p}(T_p\partial M) = T_p\partial M$.*

6.1 Orientation On The Boundary

Let $p \in \partial M$.

Inward Pointing We say $X_p \in T_pM$ is *inward pointing* if $X_p \notin T_p\partial M$ and there exists some $c : [0, \varepsilon] \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

Outward Pointing We say $X_p \in T_pM$ is *outward pointing* if $-X_p$ is inward pointing.

Proposition 6.3. *On any manifold with boundary, there exists a C^∞ outward-pointing vector field along ∂M.*

Proposition 6.4 (Induced Boundary Orientation)**.** *Let M be an oriented n-manifold with boundary. If ω is an* orientation form *(a nowhere-vanishing n-form consistent with the fxied orientation on M) and X is a C^∞ outward-pointing vector field along ∂M, then $\iota_X\omega$ is a smooth nowhere vanishing $n-1$ form on ∂M. Thus, ∂M is orientable, and $\iota_X\omega$ is called the* induced orientation*.*

Explanations on Oriented Charts Let (U, ϕ) be a chart, then $dx^1 \wedge \cdots \wedge dx^n$ is an orientation form on U. Let $X = -\frac{\partial}{\partial x^n}$, then

$$\iota_{-\frac{\partial}{\partial x^n}}(x^1 \wedge \cdots \wedge dx^n) = (-1)^n x^1 \wedge \cdots \wedge dx^{n-1}$$

is the $n-1$ orientation form on boundary. So $(U \cap \partial M, \psi = ((-1)^n x_1, ..., x_n))$ is a chart in the oriented atlas of ∂M with respect to the boundary orientation.

Thumb I don't have good intuition for how this choosing of an outward vector works, but in 3-dimensions I think of it as comparing the left and right hand rules. Depending on how my surface is oriented, choosing which way my thumb points necessarily dictates whether I am using my left hand or my right hand.

7 Integration

7.1 One Chart, One Dimension

I think I will refer to my handwritten notes for detail here. Sorry to disappoint.

7.2 Generalization

Let $\omega \in \Omega^n(M)$ be compactly supported, where M is an oriented manifold. Let $\{(U_\alpha, \phi_\alpha)\}$ be an oriented atlas, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}_\alpha$. Then each $\rho_\alpha\omega$ is compactly supported in each chart U_α, so we can make sense of

$$\int_M \rho_\alpha \omega := \int_{\rho_\alpha(x^1,\ldots,x^n)} \omega\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)\bigg|_{\phi^{-1}(x^1,\ldots,x^n)} dx^1 \cdots dx^n$$

And we can define

$$\int_M \omega := \sum_\alpha \int_M \rho_\alpha \omega$$

Proposition 7.1 (Properties of the Integral)**.** *The integral as defined above satisfies:*

1. *$\int_M \omega$ is independent of the partition of unity and the oriented atlas*
2. *The sum is finite, since $\{U_\alpha\}$ is an open cover of* $\operatorname{supp}(\omega)$
3. *$\int_{-M} \omega = -\int_M \omega$*
4. *The integral is $\mathbb{R}$-linear, i.e. for all $a \in \mathbb{R}$, ω, $\eta \in \Omega^n(M)$ compactly supported, we have that*
$$\int_M a\omega + \eta = a\int_M \omega + \int_M \eta$$
5. *Let $F : N \to M$ and let $\omega \in \Omega^n(M)$ be compactly supported, then*
$$\int_M \omega = \int_N F^*\omega$$
6. *If M can be covered by one chart U up to a measure-0 set then*
$$\int_M \omega = \int_U \omega$$

7.3 Stokes

Theorem 7.2 (Stokes' Theorem)**.** *Let M be an oriented manifolds with boundary, and let ∂M be the boundary with the induced boundary orientation. Let $\omega \in \Omega^{n-1}(M)$ be compactly supported, and let $i : \partial M \hookrightarrow M$ be the inclusion map. Then*

$$\int_M d\omega = \int_{\partial M} i^*\omega = \int_{\partial M} \omega$$