# Fractals: Are They Useful, Or Are They Just Frick-and-Frac?

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After attending a weekend conference in New York City, you finally land back in Toronto's Billy Bishop Airport. While waiting to unboard, you, an intellectual, begin to wonder about the size of the island upon which you are situated. A quick Wikipedia search tells you that the group of islands has a total area of about 820 acres, but what about the perimeter? Upon first search, Google Maps gives you a beautiful red outline of the Centre Island. You could probably just add up the total length of the red outline. Looking closer, however, you notice that the red line seems to disregard some details of the coastline. At the limit of Google Maps' satellite view zoom (See Figure 1), you find yourself ready to trace the blurry outlines of these trees to get a more accurate measure of Toronto Centre Island's coastline, instead of the beautifully smooth but disgracefully inaccurate red Google Maps line. This newly traced perimeter, however, would be longer than the red line perimeter. Any straight edge on the red outline is the shortest distance between the two endpoints of that edge; in our traced tree sketch, however, the same two endpoints will have a lot more detailed curves and turns between them - each tree, or even each branch, is accounted for. How accurate can we possible get with our measurement of the length of the perimeter of Centre Island?


Figure 1: Google Maps view of Centre Island, furthest zoom out, closer, and closest.

To answer this question, we must turn to fractals. Popularly, fractals are referred to as intricate patterns that never end, that are infinitely complex, and that are self-similar[1] across different scales[2]. Some fractals can be constructed very easily. Suppose you have an equilateral triangle. Onto the middle of each edge of this triangle, attach a smaller equilateral triangle, whose width is exactly $\frac{1}{3}$ of the width of the original edge. Repeating this process infinitely times will produce the famous fractal known as the *Koch Snowflake* (See Figure 2). If the length of each side of the first triangle is 1, the perimeter of the triangle, which we can call $P_0$, is 3. After the first round of attaching smaller triangles, the new perimeter is $P_1 = \frac{4}{3}P_0 = (\frac{4}{3})3 = 4$. After attaching another round of smaller triangles, we have $P_2 = \frac{4}{3}P_1 = \frac{4}{3}(\frac{4}{3}P_0) = (\frac{4}{3})^2 3 \approx 5.33$. Continuing this process to infinity, we observe that the exponent of the $\frac{4}{3}$ term will keep increasing, and hence the perimeter will keep increasing. One might also wonder if the area will also keep growing, by the same logic. Interestingly, we can draw a circle of radius 2 around the Koch Snowflake, but the Snowflake will never grow out of this circle, which has finite area. Therefore, the area of the Koch Snowflake will not blow up to infinity as its perimeter does. This fractal is a shape that has *infinite* perimeter but *finite* area! It does not behave like a normal 2-dimensional shape does. In order to explain this phenomenon, fractal geometry and fractal analysis use different definitions of dimension to measure fractals.

A simple version of a fractal dimension is the box-counting dimension. Start with one big square box that covers or contains the entire item that you want to measure. Then slice your box into smaller boxes of equal size, and throw away any small boxes that don't touch your original item anymore. Slice again, throw away non-intercepting boxes again, and repeat this process until your boxes are infinitely small[3]. Through making the boxes infinitely small, we won't have to worry about using

[1]Wikipedia: "In mathematics, a self-similar object is exactly or approximately similar to a part of itself (i.e. the whole has the same shape as one or more of the parts)."

[2]Definition paraphrased from *https://fractalfoundation.org/resources/what-are-fractals/*.

[3]3Blue1Brown has a beautiful animation of this process: *https://youtu.be/gB9n2gHsHN4?t=624*

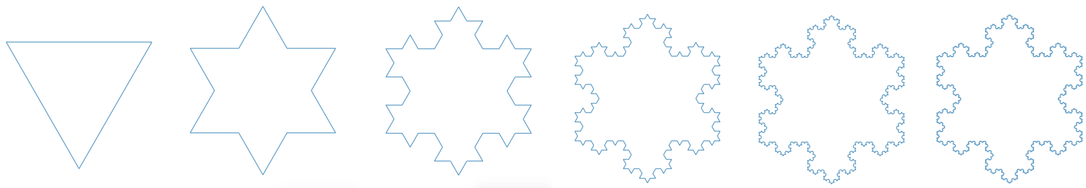

Figure 2: First 5 iterations of the Koch Snowflake.

any unit of measurement that might limit the scale and extent to which we can measure an object. To cover a 1-dimensional line segment of length 1 with boxes of width $\frac{1}{5}$, we need 5 boxes. For boxes of width $\frac{1}{2020}$, 2020 boxes. The ratio between the number of boxes left and the size of the box is consistent. With unconventional objects like fractals, however, this ratio will be inconsistent. So, as the boxes become infinitely small, we can observe rate of change of this ratio through a logarithmic equation, which represents the box-counting dimension: $D = \frac{\log(\text{number of remaining boxes})}{\log(\text{width of each box})}$. The box-counting dimension of the Koch Snowflake is known to be 1.28 - which is not surprising, as it is neither a regular 2-dimension object with finite perimeter, like a rectangle, nor a 1-dimensional object with no area, like a line. This method of measuring dimensions, in fact, can be used in many areas other than pure theoretical mathematics. For example, researchers in biology have used fractal dimension to aid their descriptions of and to supplement their data on the human brain and its cells[4], when measurements like area or volume are insufficient to describe differences between objects. Fractal geometry and fractal analysis, then, are not limited only to the perfectly recursive mathematical figures; naturally occurring objects can be treated as fractals too.

Directing our attention back to the perimeter of Toronto Centre Island, we can begin to understand why we are unable to measure the length of its outline. The more we zoom in, the more careful we have to be with throwing away our boxes, in order to tend to the details that are not apparent on the larger scale - it requires the measurement of its fractal dimension, instead of a simple length measurement. Another definition of fractals, according to Benoît Mandelbrot, "the father of fractal geometry," is anything that has a non-integer dimension. Mandelbrot explores the famous Coastline Paradox, which observes that a coastline does not have a defined length, in an essay[5] in 1967, which had profound effects on the creation of and the ensued importance of fractal analysis. He investigates various self-similar and non-self-similar curves, and suggests that length, in the case of geographical curves, is not a sufficient or appropriate measurement; in order to obtain more information about objects like a coastline, we must turn to fractal analysis.

It is a common view that fractals are only bizarre geometric objects which do not exist in the "real" world. As we have seen above, however, when we extend the definitions of dimension, fractals are a naturally occurring phenomenon. Perfectly self-similar fractals, however, is as much an over idealization of roughness as calculus is an over idealization of smoothness. In the "real" world, we cannot find or create the perfect parabola $y = x^2$, even if we get down to the microscopic or subatomic level - a physical object will never be truly smooth, and so our integral evaluation of this inaccurate curve will never actually be its actual area. This unrealistic aspect of calculus, however, does not hinder us from studying and enjoying the beauty of calculus. Just like how differential and integral calculus provide us with tools to understand and model real-life phenomena, such as the relationship between displacement and velocity in physics, idealized fractals can afford us a foundation of techniques and theories with which we can explore, quantify, and describe existing and realistic fractal shapes, such as cell structure or coastlines.

[4] T.G. Smith, Jr., G.D. Lange, W.B. Marks. *Fractal methods and results in cellular morphology - dimensions, lacunarity and multifractals.*

[5] Benoît Mandelbrot. *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension.*