Chaos: ‘Well That Escalated Quickly’

C Shijia Yu

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Imagine that you are in a situation where you could either high-five your friend or fist bump them. One version of you, version A, might have spontaneously decided to high-five them, while another version of you, B, chooses to fist bump them for no special reason. Unfortunately, due to personal hygiene issues, you end up touching your face with your hand and A accidentally contracts a highly contagious disease from your friend, while B, who didn't touch the palm of the friend's hand, does not contract the disease. You go clubbing and version A of you asymptomatically and unknowingly gives the disease to several others who drink from your cup, while version B of you does not infect anyone. You keep going to work the next week and both versions of you seem to be living the exact same life, until a week later, A and some of A's friends start showing symptoms, while B continues with a normal healthy life. Within the next week, A hears about a severe breakout of pneumonia in a local senior home, which was initially caused by A's clubbing friends who visited their grandparents.

You have just read an (awful) qualitative analogy of chaos theory and the butterfly effect. As defined by Edward Lorenz, the mathematician who first explored chaos theory, chaos is "when the present determines the future, but the approximate present does not approximately determine the future[1]." A chaotic system is deterministic, meaning that any valid input will output one and only one possible output: inputting "touch palms" outputs "contract disease," while inputting "touch knuckles" outputs "not contracting disease." Another major characteristic of chaos is that a small difference at the beginning will make a huge difference later on: the difference between a high-five and a fistbump is half a rotation of your forearm away, that's not a very big difference; the outcome, however, involves the difference between a peaceful healthy town and a pneumonia-ridden retirement home.

Consider the logistic map, a simple recurrent[2] equation that is usually used to describe population growth. We can describe the number of people with a disease by x_i, where i is the number of days since we've started counting, by this equation:

$$x_{n+1} = rx_n(1 - x_n)$$

where r is the rate of growth, and the term $(1-x_n)$ limits the population from growing exponentially fast and exploding away - realistically, a population would not keep growing forever; once the contagious disease has infected every person on earth, there can be nowhere left for the disease to grow, and so the increase in population will reach a stop. Where exactly, however, will a population growth stop? If the rate of growth is $r = 1.5$, and we start off with $x_0 = 0.5$ (Think 0.5 thousands, or 0.5 million people, instead of 0.5 people), punch this into your calculator and reiterate with your new output a few times[3], the output x_n will eventually approach the value 0.333... and stabilize; the population every year will remain at 0.333, and will never become drastically different if no conditions change. Mathematically, we say that the *sequence*[4] of outputs, x_i, *converges* to 0.333. But try $r = 3.1$, and a strange thing happens: after a while, there seems to be a loop between two values, 0.765 and 0.558. These two values just jump back and forth between each other, and the population never stabilizes to one specific capacity - instead, the series seems to "converge" to two values. Even more strangely, choosing $r = 3.5$ and iterating many times will gives us *four* values of the population x_n that we will cycle between: 0.845, 0.383, 0.827, and 0.501. The phenomenon does not stop here. As we move to a higher value of r, we start to get 8 values (at around $r = 3.544$), 16 values (at around $r = 3.564$), and so on, at a quickly escalating rate. That is, the number of

[1] https://en.wikipedia.org/wiki/Chaos_theory

[2] According to Wikipedia, "In mathematics, a recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given; each further term of the sequence or array is defined as a function of the preceding terms." This just means that I apply some function f to some value, say x_0 and get a new value, x_1, and plug the new value x_1 into the same function f and get x_2, and reuse this, etc.

[3] you can easily try this yourself by looking up "calculator" on Google. Enter 0.5 as your first equation; then, input "1.5 Ans (1−Ans)" into the calculator, and keep clicking the equals sign. The Google Calculator will automatically replace the x_n position with each of the previous answer "Ans."

[4] In case you are curious, the sequence you get with starting population 0.5 and growth rate 1.5 is: 0.5, 0.375, 0.352, 0.342, 0.338, 0.335, 0.334, 0.334, 0.334, 0.333, 0.333, 0.333, 0.333, 0.333, 0.333...

"stable" cyclic outputs will come sooner and sooner with the increase of r. The graphs[5] below in Figure 1 give a visual representation of what these cycles look like.

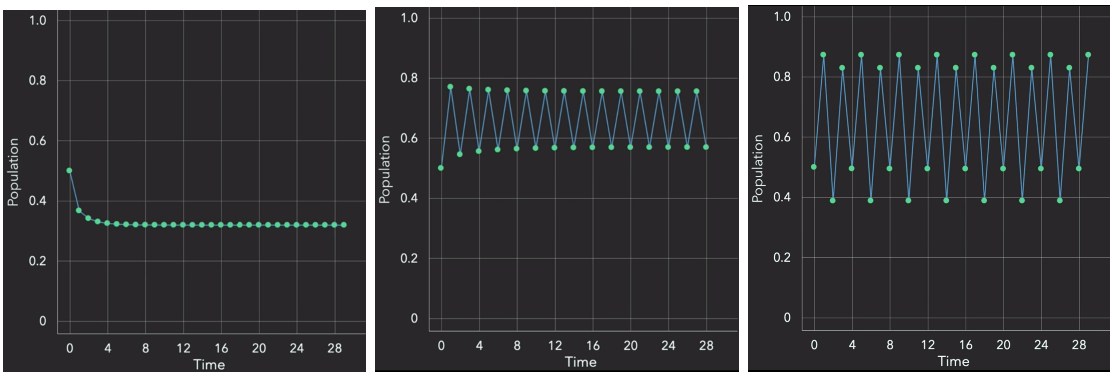

Figure 1: Behavior of Population against Time with Growth Rate $r = 1.5$, 3.1, 3.5

As we increase the value of r, we come to find that the population, after doubling its cyclic periods a few times, become messy and does not seem to have a pattern anymore. Plotting each value of r against the points to which the population x converges, we get an interesting image[6]: We can see

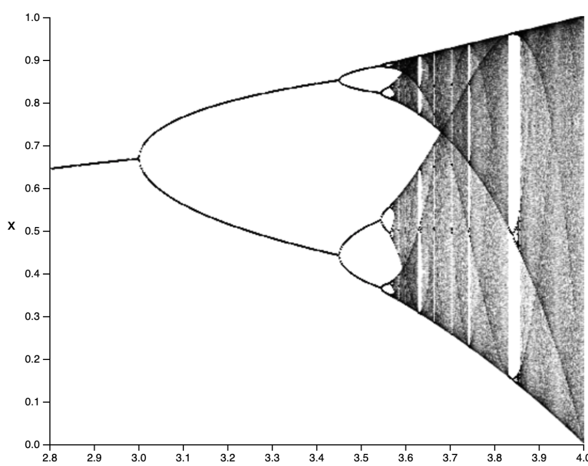

Figure 2: Points of Convergence of Populations for Varying Values of Growth Rate r

that before 3 on the horizontal axis representing r, the values of x are about midway between 0.6 and 0.7. Between $r = 3$ and $r = 3.5$, there are two defined values on the graph; between $r = 3.5$ and $r \approx 3.57$, 4 defined values. Soon enough, there are so many defined values for each t that is a tiny little bit differetn from the t before it, that our graph begins to look like it's been shaded in because there are so many points drawn on it.

Chaotic systems like this inevitably bring up issues, one being that it is difficult, if not entirely impossible, to repeat one realization of such a function. Modeling a certain population growth of a

[5] *I do not own these images!!* These are screenshots taken directly from Veritasium's YouTube video named "This equation will change how you see the world (the logistic map)" and are only meant to provide you with a brief idea, as these are *not* my images or my work!! Find the original video here: https://www.youtube.com/watch?v=ovJcsL7vyrk

[6] *I do not own this image!!* This was taken directly from the website Complexity Explorables and it is not my work!! Only used here for reference. Link to original website: https://www.complexity-explorables.org/flongs/logistic/

virus, we might find that the growth rate in one country is $r = 3.57$ and in another, 3.58. Whereas in the first country we can predict the population of the virus to settle and cycle between 4 values, the second country will have chaotic results, despite the different between the growth rates for each country is only 0.01. In fact, even if the difference of the growth rate r between the two countries are extremely extremely small, they will eventually still blow up and become drastically different within a relatively short period of time. It is therefore very difficult to recreate the same exact growth rate for 2 separate trials of population growth where the results will exactly match. On the other hand, as brilliant as the human brains are, we have in fact found very useful ways to use this chaotic and seemingly unpredictable. For example, computers use simple dynamical systems[7] with chaotic behaviours to generate pseudo-random numbers[8] because it may function perfectly as well as true random numbers[9]. Above all, the simple nature of the functions or transformations that generate such complex and chaotic behaviour is enough to convince us of the beauty of mathematics.

[7] To read more about dynamical systems, refer to my earlier paper *Dynamics: What is a Dynamical System and Why Do I Care?*.

[8] "A pseudorandom number generator (PRNG), also known as a deterministic random bit generator (DRBG), is an algorithm for generating a sequence of numbers whose properties approximate the properties of sequences of random numbers," according to Wikipedia. https://en.wikipedia.org/wiki/Pseudorandom_number_generator

[9] Check out this cool video by Veritasium on YouTube: https://www.youtube.com/watch?v=fDek6cYijxI